1. Suppose \( f : A \rightarrow B \). Define the inverse set as
\[
f^{-1}(b) = \{ a \in A \mid f(a) = b \} \quad \text{for} \ b \in B.
\]
Note that \( f^{-1}(b) \) is a set. Prove that the collection of these inverse sets
\[
\{ f^{-1}(b) \mid b \in B \}
\]
is a partition of \( A \). Hint: You need to show two properties. First, prove that for all \( a \in A \), there is some \( b \) such that \( a \in f^{-1}(b) \). Second, show that each \( a \in A \) belongs to only one set \( f^{-1}(b) \) (and hence the sets \( f^{-1}(b) \) must be disjoint).

Proof: To show that the collection of sets \( f^{-1}(b) \) is a partition,
(1) \( A \) we must show that (1) the union of all sets \( f^{-1}(b) \) forms all of \( A \), and (2) these sets are all mutually disjoint.

1. \( A = \bigcup_{b \in B} f^{-1}(b) \); (i.e., \( A = \) the union of all sets \( f^{-1}(b) \))

2. \( A \subseteq \bigcup_{b \in B} f^{-1}(b) \):

   Let \( x \in A \). Then since \( f \) is a f'n, \( \exists b \in B \) s.t. \( f(x) = b \).

   So \( x \in f^{-1}(b) \), and is thus in the union of all such sets.

   \( \bigcup_{b \in B} f^{-1}(b) \subseteq A \):

   Let \( x \in \bigcup_{b \in B} f^{-1}(b) \). By def union, \( \exists b \in B \) s.t. \( x \in f^{-1}(b) \).

   By def'n, \( f^{-1}(b) = \{ x \in A \mid f(x) = b \} \). So

   \( f^{-1}(b) \subseteq A \), and so \( x \in A \).

Since \( f \) is a f'n, \( \forall a \in A \), \( f \) maps \( a \) to a single element, call it \( b \) \in B.

So \( a \) is only in one set \( f^{-1}(b) \). And \( f^{-1}(b) \subseteq A \), so it is only made of elements of \( A \), since we showed any \( a \in A \) can only be in one set. They are mutually disjoint. \( \Box \)
2. Suppose $A$ is countable and $B$ is uncountable. Is $A \cap B$ countable? Is $A \cup B$ countable? Why?

\[ A \cap B \subseteq A, \text{ } A \text{ is countable. Therefore } A \cap B \text{ is countable because any subset of a countable set is countable.} \]

\[ B \subseteq A \cup B, \text{ } B \text{ is uncountable. Therefore } A \cup B \text{ is uncountable because any set with an uncountable subset is uncountable.} \]
3. Prove by induction that

\[ \sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3} \quad \text{for} \quad n \geq 1. \]

**Proof:**

***Base case.***

LHS. \[ \sum_{i=1}^{1} i(i+1) = 1(1+1) = 2 \]

The two sides are equal so the base case holds.

**Inductive hypothesis.** Let \( k \geq 1 \) and assume \( P(k) \) holds.

That is \[ \sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3} \]

**Inductive step.** We must prove \( P(k+1) \) holds.

\[ \sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \]

By separating terms, by applying IH, algebra

\[ = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \]

\[ = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \]

\[ = \frac{(k+1)(k+2)(k+3)}{3} \]

And this is what we wanted to show. \( \square \)
4. Define the fibonacci sequence as

\[ f_1 = 1 \]
\[ f_2 = 1 \]
\[ f_i = f_{i-1} + f_{i-2} \quad \text{for } i \geq 3. \]

Using induction, prove that \( f_{3k} \) is even for all \( k \geq 1 \) (e.g. \( f_3 \) is even, \( f_6 \) is even, etc.).

Define \( P(k) : f_{3k} \) is even.

Proof:

Basis step

Base case \( k = 1 \). Want to show \( P(1) \), i.e. that \( f_3 \) is even.

By def of sequence

\[ f_3 = f_2 + f_1 \]
\[ = 1 + 1 \]
\[ = 2 \]

2 is even because \( 2 = 2 \cdot 1 \), \( 1 \in \mathbb{Z} \), so \( P(1) \) holds.

Inductive Hypothesis

Let \( k \geq 1 \) be a fixed but arbitrary \( k \), and assume \( P(j) \) holds for all \( 1 \leq j \leq k \). That is

\[ f_{3j} \] is even.

Inductive Step: Want to show \( P(k+1) \), that is \( f_{3(k+1)} \) is even.

\[ f_{3(k+1)} = f_{3k+3} \]

By def of sequence,

\[ f_{3k+3} = f_{3k+2} + f_{3k+1} \]

by expanding subscript.

By def of sequence,

\[ f_{3k+2} = f_{3k+1} + f_{3k} \]
\[ f_{3k+1} = f_{3k} + f_{3k-1} \]

by def.

\[ f_{3k+2} = f_{3k} + 2f_{3k+1} \]

by grouping terms.

Since \( k, k-1 \leq k \), can apply inductive hypothesis, so \( 3m, n \in \mathbb{Z} \) s.t.

\[ f_{3k} = 2m + 2(2m + 2n) = 2(m + 2m + 2n) \]. Since \( (m + 2m + 2n) \in \mathbb{Z} \),

\[ f_{3k+2} \] even by def even.
5. Let $d$ and $k$ be positive integers. Define a relation $R$ on $\mathbb{Z}$ as 
\[(x, y) \in R \text{ if } d \mid (x^k - y^k),\]
Prove that $R$ is an equivalence relation.

Proof: $R$ is an equivalence relation iff and only if it is reflexive, symmetric, and transitive.

Let $d, k \in \mathbb{Z}^+$ be fixed but arbitrary positive integers.

Reflexive: [wts. $\forall x \in \mathbb{Z}$, $xRx$]

Let $x \in \mathbb{Z}$. Then $x^k = x^k$, so $x^k - x^k = 0$.

And $d \mid 0 \implies 0 = d \cdot 0$.

So $d \mid (x^k - x^k)$ so $xRx$ by def $R$. \(\square\)

Symmetric [wts $\forall x, y \in \mathbb{Z}$, if $xRy$ then $yRx$]

Let $x, y \in \mathbb{Z}$ s.t. $xRy$.

By def. of $R$, $d \mid (x^k - y^k)$.

By def. divisibility, $x^k - y^k = dl$ for some $l \in \mathbb{Z}$.

Multiplying by $(-1)$ on both sides,

$y^k - x^k = -dl$.

Since $-l \in \mathbb{Z}$, $d \mid (y^k - x^k)$ by def. divisibility.

So $yRx$ by def. $R$. \(\square\)

Transitive [wts $\forall x, y, z \in \mathbb{Z}$, if $xRy$ and $yRz$, then $xRz$]

Let $x, y, z \in \mathbb{Z}$ s.t. $xRy$ and $yRz$.

By def. $R$, $d \mid (x^k - y^k)$ and $d \mid (y^k - z^k)$.

Note that $(x^k - y^k) + (y^k - z^k) = x^k - z^k$. \(\Theta\)

By def. divisibility, $x^k - y^k = dl$ for $l \in \mathbb{Z}$ and

$y^k - z^k = mL$ for $M \in \mathbb{Z}$.

So $(x^k - y^k) + (y^k - z^k) = dl + mL = d(m + l)$. Since $m + l \in \mathbb{Z}$, $d \mid (x^k - z^k)$ by def. divisibility.

Subbing $m$ from $\Theta$, $d \mid (x^k - z^k)$.

So $xRz$ by def. $R$. \(\square\)
6. Prove that for any sets $A, B, C$

$$(A - B) \cap (A - C) = A - (B \cup C).$$

To prove equality we will prove both subset relations.

1. $$(A - B) \cap (A - C) \subseteq A - (B \cup C);$$

Let $x \in (A - B) \cap (A - C)$. Then by def intersection, $x \in A - B$ and $x \in A - C$. So $x \in A$ and $x \in B$ and $x \in C$, by def set difference. In particular $x \in A$. Since $x \in B$ and $x \in C$, $x \in B \cap C$, by def complement. So $x \in B \cap C = (B \cup C)^c$ by De Morgan's law. So $x \notin B \cup C$ by def complement. Since $x \in A$ and $x \notin B \cup C$, $x \in A - (B \cup C)$ by def set difference.

2. $A - (B \cup C) \subseteq (A - B) \cap (A - C);$  

Let $x \in A - (B \cup C)$, so $x \in A$ and $x \notin B \cup C$. So $x \in (B \cup C)^c$ so $x \notin B \cap C$ by De Morgan's law. So $x \notin B$ and $x \notin C$ by def complement. Since $x \in A$ and $x \notin B$, $x \in A - B$ by def set difference. Since $x \in A$ and $x \notin C$, $x \in A - C$ by def set difference. So $x \in (A - B) \cap (A - C)$ by def intersection.

Note: Instead of using De Morgan's laws for sets, you could have used the following logic in the appropriate places in

1. Since $x \notin B$ and $x \notin C$, then $x$ is not in $B$ or $C$ (by logical De Morgan's law), so $x \notin B \cup C$ by def union.

2. Since $x \notin B \cup C$, i.e., $x$ is not in $B$ or $C$, so $x$ is not in $B$ and $x$ is not in $C$ by logical De Morgan's law, so $x \notin B$ and $x \notin C$. 


7. How many elements does $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ have? ($\mathcal{P}(A)$ denotes the powerset of $A$).

- $\emptyset$ has 0 elements.
- $\mathcal{P}(\emptyset) = \{\emptyset, \{\emptyset\}\}$ has one element.
- The powerset of a set containing $n$ elements has $2^n$ elements.
  [This is from class.

Since $\mathcal{P}(\emptyset)$ contains 1 elt, $\mathcal{P}(\mathcal{P}(\emptyset))$ contains $2^1 = 2$ els.

Since $\mathcal{P}(\mathcal{P}(\emptyset))$ contains 2 els, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ contains

$2^2 = 4$ elements.

Note:

<table>
<thead>
<tr>
<th>Set name</th>
<th>Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>${\emptyset}$</td>
</tr>
<tr>
<td>$\mathcal{P}(\emptyset)$</td>
<td>${\emptyset, {\emptyset}}$</td>
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<tr>
<td>$\mathcal{P}(\mathcal{P}(\emptyset))$</td>
<td>${\emptyset, {\emptyset, {\emptyset}}}$</td>
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<tr>
<td>$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$</td>
<td>${\emptyset, {\emptyset, {\emptyset, {\emptyset}}}, {\emptyset, {\emptyset, {\emptyset}}} }$</td>
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</tbody>
</table>
8. A relation $R$ on $A$ is circular if for all $x, y, z \in A$, $xRy$ and $yRz$ implies $zRx$. Show that a reflexive circular relation is an equivalence relation.

Proof.

reflexive: This is given, as we are told the relation is a "reflexive circular relation."

symmetric: \[ \forall x, y \in A, \text{ if } xRy \text{ then } yRx \]

Let $x, y \in A$ s.t. $xRy$. From reflexive, we know that $yRy$. From definition of circular, we have that $yRx$. \\
(Let $y = z$ in the definition above so $xRy$ and $yRy \rightarrow yRx$)

transitive: \[ \forall x, y, z \in A, \text{ if } xRy \text{ and } yRz \text{ then } xRz \]

Let $x, y, z \in A$ s.t. $xRy$ and $yRz$. By definition circular $zRx$. But we have shown above that $R$ is symmetric, so $xRz$. $\blacksquare$
9. Suppose that \( f : A \rightarrow B \), \( g : B \rightarrow C \) are both onto. Prove that \( g \circ f \) is onto.

\[
\text{proof: Let } f : A \rightarrow B, \ g : B \rightarrow C
\]
both be onto.

\[
\text{[wts g \circ f onto, ie } \forall c \in C, \ \exists a \in A \text{ st. } (g \circ f)(a) = c] \]

Let \( c \in C \). Since \( g \) is onto, \( \exists b \in B \) st. \( g(b) = c \). Since \( f \) is onto, \( \exists a \in A \) st. \( f(a) = b \). So \( g(f(a)) = g(b) = c \).

Since \( g(f(a)) = (g \circ f)(a) \) by def composition of funs, we have given an \( a \) s.t. \( (g \circ f)(a) = c \). There sone \( g \circ f \) is onto by definition onto. \( \square \)