Image Formation, Cameras

Computer Vision I

CSE 252A

Lecture 4

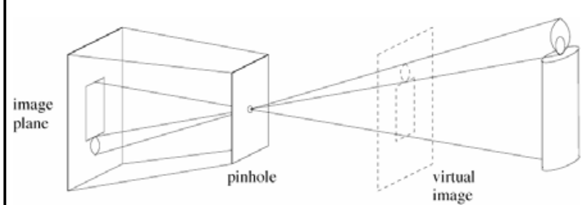
Announcements

- Read Chapters 1 & 2 of Forsyth & Ponce
- Homework 1 – will be on web page in next day or two, due Jan. 31
- Office Hours
  - Kriegman: Wed 1:00-2:30
  - Neil: Tues, 11:00-12:00, EBU3B, B240a

Last lecture in a nutshell

Pinhole Camera: Perspective projection

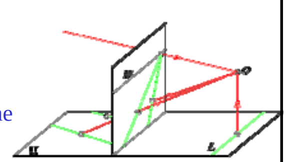
- Abstract camera model - box with a small hole in it



 Forsyth&Ponce

Geometric properties of projection

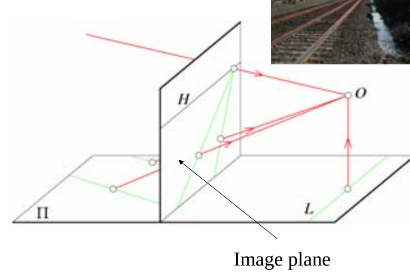
- Points go to points
- Lines go to lines
- Planes go to whole image or half-plane
- Polygons go to polygons
- Angles & distances not preserved



- Degenerate cases:
  - line through focal point yields point
  - plane through focal point yields line

Parallel lines meet in the image

- vanishing point



Projective Geometry

- Axioms of Projective Plane
  1. Every two distinct points define a line
  2. Every two distinct lines define a point (intersect at a point)
  3. There exists three points, A,B,C such that C does not lie on the line defined by A and B.
- Different than Euclidean (affine) geometry
- Projective plane is "bigger" than affine plane – includes "line at infinity"

Homogenous Coordinates & the camera matrix

Homogenous Coordinates

– Euclidean -> Homogenous: (x, y) -> k (x,y,1)

– Homogenous -> Euclidean: (x, y, z) -> (x/z, y/z)

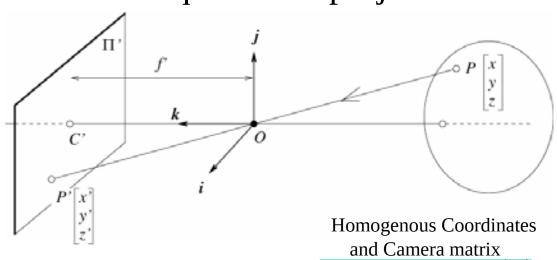
– Projective transformation

– 3 x 3 linear transformation of homogenous coordinates

– Points map to points, lines map to lines

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{21} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The equation of projection



Cartesian coordinates:

$$(x,y,z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

Homogenous Coordinates and Camera matrix

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

- Perspective

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{f}{z} \begin{bmatrix} x \\ y \end{bmatrix}$$

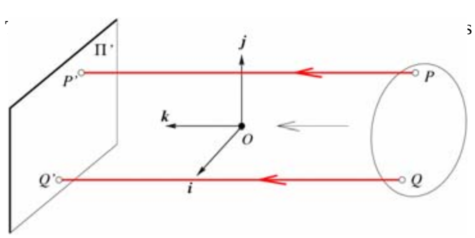
- Assume that $f=1$, and perform a Taylor series expansion about (x_0, y_0, z_0)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{z_0}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{z_0^2}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}(z-z_0) + \frac{1}{z_0}\begin{bmatrix} 1 \\ 0 \end{bmatrix}(x-x_0)$$
$$+ \frac{1}{z_0}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(y-y_0) + \frac{1}{2}\frac{2}{z_0^3}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}(z-z_0)^2 + \cdots$$

- Dropping higher order terms and regrouping.

$$\begin{bmatrix} u \\ v \end{bmatrix} \approx \frac{1}{z_0}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} 1/z_0 & 0 & -x_0/z_0^2 \\ 0 & 1/z_0 & -y_0/z_0^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{Ap} + \mathbf{b}$$

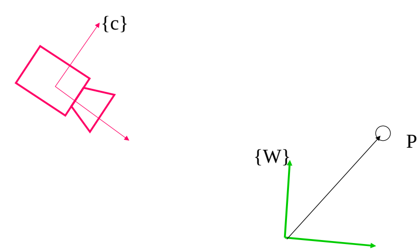
Orthographic projection



$$\begin{cases} x' = x \\ y' = y \end{cases}$$

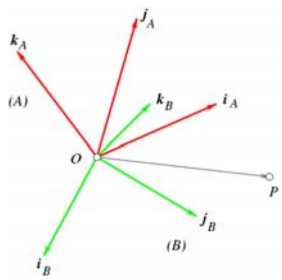
$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

What if camera coordinate system differs from object coordinate system



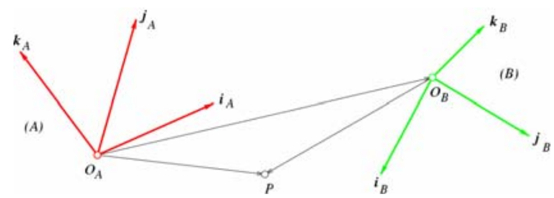
Translation & Rotation between Coordinate Systems

Rotation Matrix



$$
{}^B_A R = \begin{bmatrix} \mathbf{i}_A.\mathbf{i}_B & \mathbf{j}_A.\mathbf{i}_B & \mathbf{k}_A.\mathbf{i}_B \\ \mathbf{i}_A.\mathbf{j}_B & \mathbf{j}_A.\mathbf{j}_B & \mathbf{k}_A.\mathbf{j}_B \\ \mathbf{i}_A.\mathbf{k}_B & \mathbf{j}_A.\mathbf{k}_B & \mathbf{k}_A.\mathbf{k}_B \end{bmatrix} = \begin{bmatrix} {}^A\mathbf{i}_B^T \\ {}^A\mathbf{j}_B^T \\ {}^A\mathbf{k}_B^T \end{bmatrix} = \begin{bmatrix} {}^B\mathbf{i}_A & {}^B\mathbf{j}_A & {}^B\mathbf{k}_A \end{bmatrix}
$$

Coordinate Changes: Rigid Transformations



$$
{}^B P = {}^B_A R \, {}^A P + {}^B O_A
$$

Another Digression

Rotation Matrices

SO(3)

Some points about SO(n)

- $SO(n) = \{ R \in \Re^{nxn} : R^T R = I, \det(R) = 1\}$
  - SO(2): rotation matrices in plane $\Re^2$
  - SO(3): rotation matrices in 3-space $\Re^3$
- $R^{-1} = R^T$
- Bounded $R_{i,j} \in [-1, +1]$
- FYI, if det(R)= -1, there would be a reflection

  E.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

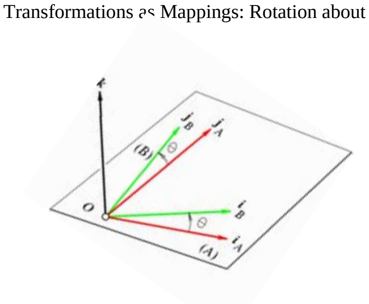
Some more points about SO(n)

- $SO(n) = \{ R \in \Re^{nxn} : R^T R = I, \det(R) = 1\}$
- Forms a Group under matrix product operation:
  - Identity
  - Inverse
  - Associative
  - Closure
- Rotations are not commutative in general
- Manifold of dimension n(n-1)/2
  - Dim(SO(2)) = 1
  - Dim(SO(3)) = 3

SO(3)

- Parameterizations of SO(3)
- 3-D manifold, so between 3 parameters and 2n parameters (Whitney's Embedding Thm.)
  - Roll-Pitch-Yaw
  - Euler Angles
  - Axis Angle (Rodrigues formula)
  - Cayley's formula
  - Matrix Exponential
  - Quaternions (four parameters + one constraint)

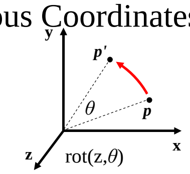
Rigid Transformations as Mappings: Rotation about the **k** Axis



$$ {}^F P' = \mathcal{R}\, {}^F P, \quad \text{where} \quad \mathcal{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{rot}(k,\theta) $$

Rotation: Homogenous Coordinates

- About z axis



$$ \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} $$

Rotation

- About x axis:

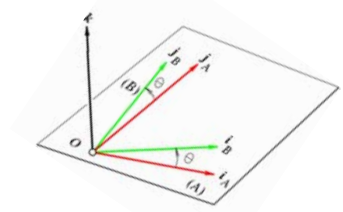
$$ \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} $$

- About y axis:

$$ \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 1 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} $$

Roll-Pitch-Yaw

$$ R = rot(\hat{i},\alpha)rot(\hat{j},\beta)rot(\hat{k},\varphi) $$

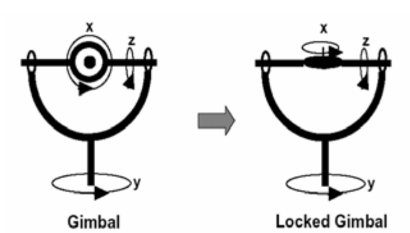


Euler Angles

$$ R = rot(\hat{k}'',\alpha)rot(\hat{j}',\beta)rot(\hat{k},\varphi) $$

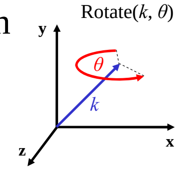
A Gimbal

- ***Hardware implementation of Euler angles (used for mounting gyroscopes and globes)***



Rotation

- About (k_x, k_y, k_z), a unit vector on an arbitrary axis (Rodrigues Formula)



$$ \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} k_x k_x(1-c)+c & k_z k_x(1-c)-k_z s & k_x k_z(1-c)+k_y s & 0 \\ k_y k_x(1-c)+k_z s & k_z k_x(1-c)+c & k_y k_z(1-c)-k_x s & 0 \\ k_z k_x(1-c)-k_y s & k_z k_x(1-c)-k_x s & k_z k_z(1-c)+c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} $$

where $c = \cos\theta$ & $s = \sin\theta$

Quaternions

$q = (a, \boldsymbol{\alpha})$ — q is a quaternion (generalization of imaginary numbers)

$a \in$ R is its real part (cos angle of rotation)

$\boldsymbol{\alpha} \in$ R^3 is its imaginary part (axis of rotation)

Operations on quaternions:

- Sum of quaternions: $(a, \boldsymbol{\alpha}) + (b, \boldsymbol{\beta}) \equiv ((a+b), (\boldsymbol{\alpha}+\boldsymbol{\beta}))$
- Multiplication by a scalar: $\lambda\,(a, \boldsymbol{\alpha}) \equiv (\lambda a, \lambda \boldsymbol{\alpha})$
- Quaternion product:

$$(a, \boldsymbol{\alpha})\,(b, \boldsymbol{\beta}) \equiv ((a\,b - \boldsymbol{\alpha} \cdot \boldsymbol{\beta}),\ (a\,\boldsymbol{\beta} + \mathrm{b}\,\boldsymbol{\alpha} + \boldsymbol{\alpha} \times \boldsymbol{\beta}))$$

- Conjugate: $q = (a, \boldsymbol{\alpha})$ $\quad q' \equiv (a, -\boldsymbol{\alpha})$
- Norm: $|\,q\,|^2 \equiv q\,q' = q'\,q = a^2 + |\,\boldsymbol{\alpha}\,|^2$

Unit Quaternions and Rotations

- Let R denote the rotation of angle θ about the unit vector $\boldsymbol{u}$.
- Define unit quaternion $q = (\cos \theta/2,\ \sin \theta/2\ \boldsymbol{u})$.
- Note $|q| = 1$ (i.e., q lies on unit sphere for any $\boldsymbol{u}$ and θ.
- Then for any vector $\boldsymbol{\alpha}$,
  $R\,\boldsymbol{\alpha}$ = imaginary($q\,\boldsymbol{\alpha}^*\,q'$)
  where $\boldsymbol{\alpha}^* = (0,\ \boldsymbol{\alpha})$
- q and –q define the same rotation matrix.

If $q = (a, (b, c, d)^T)$ is a unit quaternion, the corresponding rotation matrix is:

$$\mathcal{R} = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & a^2-b^2-c^2+d^2 \end{pmatrix}$$

Block Matrix Multiplication

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

What is AB ?

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Homogeneous Representation of Rigid Transformations

$$\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R\,{}^{A}P + {}^{B}O_{A} \\ 1 \end{bmatrix} = {}^{B}_{A}T \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$$

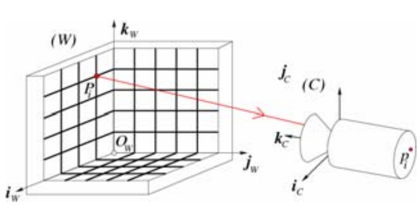
Camera parameters

- Issue
  - camera may not be at the origin, looking down the z-axis
    - extrinsic parameters
  - one unit in camera coordinates may not be the same as one unit in world coordinates
    - intrinsic parameters - focal length, principal point, aspect ratio, angle between axes, etc.

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{intrinsic parameters} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \text{Transformation} \\ \text{representing} \\ \text{extrinsic parameters} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

3 x 3 4 x 4

Camera Calibration



Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$, estimate intrinsic and extrinsic camera parameters

- See Text book for how to do it.