2. 3 pts. How many distinct combinations of inputs are there for an $n$-input Boolean function?
   (a) $2^n$
   (b) $2^{2n}$
   (c) $n^2$
   (d) $n^{2^n}$
   (e) $n$

3. 3 pts. How many distinct $n$-input Boolean functions are there?
   (a) $2^n$
   (b) $2^{2^n}$
   (c) $n^2$
   (d) $n^{2^n}$
   (e) $n$
4. 5 pts. Circle each of the following sets that are uncountable:
   (a) Mersenne primes
   (b) Real numbers
   (c) Irrational numbers
   (d) The set of students who will score 200% on this exam
   (e) The set of all subsets of rational numbers

5. 10 pts. Use twos-complement 6-bit arithmetic to compute -5 - 22. Show your work.

6. 7 pts. Convert 57260563_g to hexadecimal.

7. 15 pts. Design a boolean function (using the operators ∧, ∨, and ¬) equal to $S$ as defined by the following truth table:

\[
\begin{array}{ccc|c}
p & q & r & S \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
\end{array}
\]
8. 15 pts. Prove that if $a$, $b$, and $c$ are integers for which $a|(b + c)$ and $a|b$, then $a|c$.

9. 20 pts. Given:
   - predicates $I(x) = "x is wise"
   - $E(x,y) = "x knows y"$, and a domain $D = \{a,b,c\}$.
   - the truth set of $I(x)$ is $\{a\}$, and the truth set of $E(x,y)$ is $\{(a,a),(b,b),(c,c),(a,c),(a,b)\}$.
   - [Remember that the truth set of a predicate is the set of elements in the domain that make the predicate true.]

   Answer the following questions:
   (a) What is the truth value of $\forall x \in D \exists y \in D : I(y) \iff E(x,y)$?

(b) What is the truth value of $\exists y \in D : \forall x \in D : \sim E(x,y) \iff \sim I(x)$?
9. 20 pts. Given:
• predicates $I(x) =$ “$x$ is wise” and $E(x,y) =$ “$x$ knows $y$”, and a
domain $P = \{a, b, c\}$, and
• the truth set of $I(x)$ is $\{a\}$, and
• the truth set of $E(x,y)$ is $\{(a,a), (b,b), (c,c), (a,c), (a,b)\}$. [Remember that the truth set of a predicate is the set of elements in the domain that make the predicate true.]

(d) Convert the following from predicate logic to English: $\exists x \in P \forall y \in P : E(x,y) \implies I(y)$.

9. 20 pts. Given:
• predicates $I(x) =$ “$x$ is wise” and $E(x,y) =$ “$x$ knows $y$”, and a
domain $P = \{a, b, c\}$, and
• the truth set of $I(x)$ is $\{a\}$, and
• the truth set of $E(x,y)$ is $\{(a,a), (b,b), (c,c), (a,c), (a,b)\}$. [Remember that the truth set of a predicate is the set of elements in the domain that make the predicate true.]

(e) Convert the following from English to predicate logic: anyone who knows everybody is wise.

10. 20 pts. Draw a combinatorial circuit containing only 1-input not and 2-input or gates that is equal to the following Boolean function (Note: make the circuit as simple as possible):

11. 15 pts. Extra credit. Prove the statement if true, otherwise find a counterexample:
Given $x \neq 0$ is rational, if $y$ is irrational, then the product $xy$ is irrational.
Definitions

**Set**: Contains zero or more elements, repetitions ignored, order doesn’t matter
- Example with no elements
  - \(\{\}\) (alternative syntax: \(\emptyset\))
- Example with three elements
  - \(\{\text{john, sally, harry}\}\) = \(\{\text{harry, sally, john}\}\)
- Example with infinite elements
  - \(\{x : x\text{ a positive integer}\}\)
- What about: \(\{\emptyset\}\)

Ordered set: Contains zero or more elements, repetitions **not** allowed, order **does** matter (not often used)
- \((\text{john, sally, harry})\)
- \((\text{sally, harry, john})\)

Ordered list: Contains zero or more elements, repetitions allowed, order **does** matter (used often)
- Ordered 4-tuple
  - \((\text{john, sally, harry, salty})\)
- Ordered pair
  - \((x, y)\)
- Ordered triple
  - \((x, y, z)\)

Set notation

- \(x \in S\)
  - \(x\) is an element of \(S\)
- \(S \subseteq T\)
  - \(S\) is a subset of \(T\):
    - \(\forall x, x \in S \rightarrow x \in T\)
- \(S \subset T\)
  - \(S\) is a proper subset of \(T\):
    - \(S \subset T \land (\exists x : x \in T \land x \notin S)\)
- \(S = T\)
  - \(S\) is equal to \(T\): \(S \subset T\) and \(T \subset S\)
- \(|S|\)
  - The cardinality (size) of set \(S\): the number of elements of \(S\)

\(k\)-set
- A set with cardinality \(k\)
- \(\{x : x > 1\}\) (or \(\{x | x > 1\}\))
- Defines the set by stating the properties shared by the elements

Set Operations

- \(S \cup T\)
  - Union: \(\{x : x \in S \lor x \in T\}\)
- \(S \cap T\)
  - Intersection: \(\{x : x \in S \land x \in T\}\)
- \(S - T\) (or \(S \setminus T\))
  - Difference: \(\{x : x \in S \land x \notin T\}\)

Often we have some universe \(U\), in mind

- \(S^C\) (or \(\sim S\))
  - Complement of \(S\): \(U - S\)
- \(S \oplus T\)
  - Symmetric difference: \((S - T) \cup (T - S)\)
- \(S \times T\)
  - (Cartesian) product: \((x, y) : x \in S \land y \in T\)

Venn Diagrams

- Rectangle is universe
- Circle (or ellipse) for each set
- Overlap represents intersection

Example:
- \(A \cap B\)
  - \(A \cap B^C\)
Proof Using Venn Diagrams

Proving LHS=RHS (or LHS≠RHS)
- Create general Venn diagram for given sets
- (optional) Number each region
  - (Given k sets, how many regions?)
- Shade in regions for LHS
- Shade in regions for RHS
- Compare

Example Proof
Prove \((A \cap B)^c = A^c \cup B^c\)

Proof Using Elements

To prove LHS=RHS
- Prove LHS ⊆ RHS
- Prove RHS ⊆ LHS

To prove LHS ⊆ RHS
- If \(x \in\) LHS, then ...
- Thus, \(x \in\) RHS
- This shows that LHS ⊆ RHS

Example: prove \((A \cap B)^c = A^c \cup B^c\)
- If \(x \in (A \cap B)^c\), then \(x \notin (A \cap B)\)
- Thus, \(x \in A^c\) or \(x \in B\)
- So, \(x \in A^c\) or \(x \in B^c\)
- Hence, \(x \in A^c \cup B^c\)
- This shows that \((A \cap B)^c \subseteq A^c \cup B^c\)

Algebraic Rules for Sets

<table>
<thead>
<tr>
<th>Rule</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associative</td>
<td>(A \cup (B \cup C) = (A \cup B) \cup C) (\quad A \cap (B \cap C) = (A \cap B) \cap C)</td>
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<tr>
<td>Distributive</td>
<td>(A \cup (B \cap C) = (A \cup B) \cap (A \cup C)) (\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C))</td>
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<tr>
<td>DeMorgan's</td>
<td>((A \cup B)^c = A^c \cap B^c) (\quad (A \cap B)^c = A^c \cup B^c)</td>
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<tr>
<td>Idempotent</td>
<td>(A \cup A = A) (\quad A \cap A = A)</td>
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<tr>
<td>Double Negation</td>
<td>((A^c)^c = A)</td>
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<tr>
<td>Absorption</td>
<td>(A \cup (A \cap B) = A) (\quad A \cap (A \cup B) = A)</td>
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<tr>
<td>Commutative</td>
<td>(A \cup B = B \cup A) (\quad A \cap B = B \cap A)</td>
</tr>
<tr>
<td>Bound</td>
<td>(A \cup \emptyset = A) (\quad A \cap \emptyset = \emptyset)</td>
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<tr>
<td>Negation</td>
<td>(A \cup A^c = \Omega) (\quad A \cap A^c = \emptyset)</td>
</tr>
</tbody>
</table>
Proof Using Algebraic Rules

\[ A = (B^c \cup A) \cap (A \cup B) \]

Tabular Method

\[ A = (B^c \cup A) \cap (A \cup B) \]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>B^c</th>
<th>B^c \cup A</th>
<th>A \cup B</th>
<th>(B^c \cup A) \cap (A \cup B)</th>
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Tabular Method vs. Venn Diagram