Predicates

Grammatical origin

Alternate definitions of a predicate:
- A sentence with a finite number of variables. Replacing variables with specific values yields a statement. The domain is the set of all possible values to substitute.
- A function whose codomain is (true/false) statements.

Example:
- \( x^2 > 1 \)
  - Domain?
  - Example statements?
- “Boy x likes girl y” (alternate syntax: )
  - Domain?
  - Example statements?

Important Sets

\( Z \) (Zahl = Number)

\( \mathbb{Z}^+ \)

\( N \) (Natural)
- No general agreement on whether it includes 0

\( Q \) (Quotient)

\( R \) (Real)

\( P \) (Prime)

Truth Set of a Predicate, \( P(x) \)

The set of all elements in the domain that make \( P(x) \) true

Denoted:
- \( \{x \in D: P(x)\} \)

Example:
- What is the truth set of \( x \) is a factor of 8? 
  - If \( D \) is \{0, 1, 2\}? 
  - If \( D \) is in \{0, 1, 2, 3, …\} (\( \mathbb{N} \))?
  - If \( D \) is in \{…, -3, -2, -1, 0, 1, 2, 3, …\} (\( \mathbb{Z} \))?
Quantifiers

Universal
- \( \forall x \in D \, S(x) \)

Existential
- \( \exists x \in D \, S(x) \)

Common to drop \( \in D \) if it is clear from context

Examples
Let \( L(x, y) \) be the predicate “Boy x likes girl y”

- \( \exists y : \exists x : L(x, y) \)
- \( \exists y : \forall x \, L(x, y) \)
- \( \forall x \, \exists y : L(x, y) \)
- \( \forall y \, L(“Jose”, y) \)

Negation and Quantifiers

\( \neg (\exists x : P(x)) \leftrightarrow \forall x \, \neg P(x) \)

\( \neg (\forall x : P(x)) \leftrightarrow \exists x : \neg P(x) \)

Examples

\( L(x, y) = “x \text{ likes } y” \) Domain: all people in my son’s 1st grade class
- Everyone likes Jill
- Nobody likes Jill
- Not everybody likes Jill
- There exists a person that nobody likes
- Nobody dislikes everybody
- Everybody likes him/herself
- \( \forall x \, \forall y \, \forall z \, (L(x, y) \land L(y, z) \rightarrow L(x, z)) \)
- \( \forall x \, \forall y \, (L(x, y) \rightarrow L(y, x)) \)
Converting to Predicates

If a number is a natural number, it is an integer

The square of any real number greater than 2 is greater than 4

Quantifiers and Adversaries

Statements involving quantifiers can be viewed as a game
- Two players
  - You, trying to prove the statement is true
  - An adversary, trying to prove the statement is false
- Work from left to right through the quantifiers
  - ∀y: the adversary selects an y (think of the adversary as choosing the worst possible y).
    That y remains bound the rest of the game
  - ∃x: you select an x that will satisfy the statement. That x remains bound the rest of the game

Example:
- ∀x∈Z, ∃y∈Z: 2x = y
- ∃x∈Z: ∀y∈Z, xy = 0
- ∀y∈Z, ∃x∈Q: x≠0→xy = 1

Limit

The limit of a sequence a is L means that a gets arbitrarily close to L

\[ \lim_{n \to \infty} a_n = L \]

∀ε ∈ R⁺, ∃n₀ ∈ N : ∀n ∈ N, n ≥ n₀ → L − ε ≤ aₙ ≤ L + ε

Order of Growth of a Function

We say that T(n) has an asymptotic upper bound of f(n) if:

∃n₀ ∈ N : ∃c ∈ R⁺ : ∀n ∈ Z, n ≥ n₀ → 0 ≤ T(n) ≤ cf(n)
Proving Quantified Statements

Prove universal statement: \( \forall x \in D, P(x) \rightarrow Q(x) \)
- Exhaustive enumeration
- Generalizing from the generic particular
  - "Suppose \( x \) is in \( D \) and \( P(x) \)"
  - …
  - Therefore \( Q(x) \)
- Example: The difference of two odd numbers is even

Prove existential statement: \( \exists x \in D: P(x) \)
- Constructive proof
  - Display an \( x \)
  - Give a set of directions for finding \( x \)
- Nonconstructive proof
  - Proof by contradiction (assume non-existence and show a contradiction)
  - Show \( x \) must exist
- Example:
  - In a group of 367 people, at least two share a birthday

Disproving Quantified Statements

Disprove universal statement: \( \forall x \in D, P(x) \rightarrow Q(x) \)
- Counterexample
  - Show an \( x \) in \( D \) where \( P(x) \), but not \( Q(x) \)
- Example: All primes are of the form \( 2^n - 1 \)

Disprove existential statement: \( \exists x \in D: P(x) \)
- Equivalent to:
  - Or, alternatively,
    - Therefore, best bet is generalizing from the generic particular.
- Example: There exists a prime which can be written as the square of an integer > 1