Ordered Subset

If \((S, \preceq)\) is a poset, then,
- Take \(X \subseteq S\)
- Take \(\preceq_X = \{a \preceq b : a, b \in X\}\)

\((X, \preceq_X)\) is a poset

We would normally write: \(X\) is an ordered subset of \(S\)
- Assume the order relation of \(X\) is the relation induced by the order relation of \(S\)

Example: given \((\mathbb{Z}, |)\) take \(\{1, 2, 3, 4, 5, 6, 7, 8\} \subseteq \mathbb{Z}\)

Product Set

Given two posets \((S, \preceq_S)\) and \((T, \preceq_T)\), we define the product set \((R, \preceq_R)\)
- \(R = S \times T\)
- \((s, t) \preceq_R (s', t')\) iff \(s \preceq_S s'\) and \(t \preceq_T t'\)

The product set is a poset

Example
- \((\mathbb{N} \times \mathbb{N}, \leq)\)

Linearly Ordered Sets

If \((S, \preceq)\) is a poset, then \(a\) and \(b\) in \(S\) are comparable iff
- \(a \preceq b\) or \(b \preceq a\)

If \((S, \preceq)\) is a poset, and each pair of elements of \(S\) are comparable, then \(S\) is a linearly ordered set (or totally ordered set)
- \(\preceq\) is a total ordering of \(S\)
- \(S\) is called a chain

Examples
- \((\mathbb{N}, \leq)\)
- \((\mathbb{N}, |)\)
- \(\{(1, 2, 32, 8, 4), |\}\)
Maximal, Minimal, Least, Greatest

Given a poset \((S, \leq)\), and an element \(e\) in \(S\):
- \(e\) is **minimal** if there is no preceding element of \(S\) (other than \(e\))
- \(e\) is **maximal** if \(e\) precedes no element of \(S\) (other than \(e\))
- \(e\) is **least** if it precedes all elements of \(S\)
- \(e\) is **greatest** if it is preceded by all elements of \(S\)

Upper and lower bounds

Given a poset \((S, \leq)\) and a subset of elements \(A \subseteq S\), \(x \in S\) is:
- an **upper bound** of \(A\) if for all \(y \in A\), \(y \leq x\)
  - Only some subsets have upper bounds
- a **lower bound** of \(A\) if for all \(y \in A\), \(x \leq y\)
  - Only some subsets have lower bounds
- a **least upper bound** of \(A\) if it is an upper bound of \(A\) and less than all other upper bounds of \(A\)
- a **greater lower bound** of \(A\) if it is a lower bound of \(A\) and greater than all other lower bounds of \(A\)

**Example:**
- \(\{3, 7, 14, 21, 40, 42\}\)

Lattice

A poset \((S, \leq)\) is a **lattice** if for pair of elements in \(S\) has a least upper bound (join) and a greatest lower bound (meet)
- Haase diagram looks like a physical lattice

**Example:**
- \(\{N, \leq\}\)
- \(\{S, \subseteq\}\)
- \(\{\mathbb{Z^+}, |\}\)
Example Topological Sort

Topological sort of prerequisites

- 8A
- 8B: requires 8A
- 12: requires 8B,
- 20: requires 12
- 21: requires 20
- 30: requires 12, 20
- 100: requires 12, 21
- 101: requires 12, 21, 100
- 105: requires 12, 21
- 120: requires 100, 101, 141
- 131A: requires 30, 100, 105
- 131B: requires 30, 100, 105, 131A
- 140: requires 20, 30
- 141: requires 140