### Sequence Cagematch

Given two sequences $a_n$ and $b_n$, which grows faster?

- Polynomial of degree $n$ versus polynomial of degree $n-1$?
- Exponential versus polynomial?
- Polynomial versus logarithmic?

### Infinite Series

Our approach will **not** be based on Bender. You are not responsible for IS section 3

- But, you are responsible for this lecture!

### Identifying a Sequence

The Online Encyclopedia of Integer Sequences


Enter part of a sequence:

- 0 1 3 6 10 15 21

Get back a list of known related sequences

<table>
<thead>
<tr>
<th>ID Number:</th>
<th>A000217 (formerly M2535 and N1002)</th>
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**Sequence:**

0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431

**Name:** Triangular numbers: $a(n) = \binom{n+1}{2} = n(n+1)/2 = 0+1+2+...+n$.

**Comments:**

- Number of edges in complete graph of order $n$, $K_n$.
- Number of legal ways to insert a pair of parentheses in a string of $n$ letters. E.g., there are 4 ways for these letters: (ab), (a)b, (ab), (a)b. (It doesn't make sense for the parentheses to overlap because the parentheses are adjacent.) E.g., comment:
  - For $n > 1$: $a(n) = \binom{n+1}{2}$ is also the genus of a noninterior curve of degree $n+1$ like the Fermat curve $x^{n+1} + y^{n+1} = 1$. - Ahmed Fares (ahmedface(AT)my_deja.com), Feb 21 2001
  - $a(n)$ is the number of ways in which $n+2$ can be written as a sum of three positive integer if repetitions differing in the order of the terms are considered to be different. In other words, $a(n)$ is the number of positive integral solutions of the equation $x + y + z = n+2$. - Amarnath Murthy (amarnath.murthy(AT)my-yahoo.com), Apr 22 2001
Paradoxes with Infinite Sums

\[ S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \]

\[ S = \text{______________} \]

\[ T = 1 + 2 + 4 + 8 + 16 + \ldots \]

\[ T = \text{______________} \]

Examples

\[ \sum_{k \geq 0} x^k = \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} \]

- if \(0 \leq x < 1:\)
  - if \(x \geq 1:\)

\[ S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \]

\[ T = 1 + 2 + 4 + 8 + 16 + \ldots = \]

Bound of \( \sum_{k \in K} a_k \)

Assume each \(a_k\) is non-negative

\( K \) may be infinite

If there exists a constant \(A\) such that, for all finite subsets \(F \subset K:\)

\[ \sum_{k \in F} a_k \leq A \]

- Then, we say \( \sum_{k \in K} a_k \) converges to the least such \(A, A_{\min}\)
- If there is no such \(A\), then we say the sum diverges to infinity.

If \(K\) is the set of non-negative integers:

- \[ \sum_{k \in K} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k \]

Negative Terms in Infinite Series

Three different answers

\[ \sum_{k \geq 0} (-1)^k \]

- \(1 + -1 + 1 + -1 + \ldots\)
- \(1 + -1 + 1 + -1 + \ldots\)

- Use formula for Geometric series
Negative Terms in Infinite Series

Two different answers

$$\sum_{k \geq 0} \frac{1}{k+1} + \sum_{k < 0} \frac{1}{k-1}$$

- Sum of n innermost parentheses =

$$\cdots + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- Sum of n innermost parentheses =

$$\cdots + \frac{1}{4} - \frac{1}{3} - \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- Sum of n innermost parentheses =

$$\cdots + \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- Sum of n innermost parentheses =

Infinite Sums with Positive and Negative Terms

Break apart sums into positive terms and negative terms

- Define $x^+$ and $x^-:
  - If $x \geq 0$, $x^+ = x$, $x^- = 0$
  - If $x < 0$, $x^+ = 0$, $x^- = -x$

Let $A^+ = \sum a_k^+$ and $A^- = \sum a_k^-$

- If $A^+$ and $A^-$ are finite
  - Sum converges absolutely to $A^+ - A^-$

- If $A^+$ is infinite, but $A^-$ is finite:
  - Sum diverges to infinity

- If $A^+$ is finite, but $A^-$ is infinite:
  - Sum diverges to negative infinity

- If $A^+$ and $A^-$ are infinite
  - Sum is undefined

Conditional Convergence

A sum where $\sum a_n$ converges, but $\sum |a_n|$ doesn’t

- Example (alternating harmonic series)
  - $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$

- Kind of wacky, because rearranging the order of terms in the series can yield different values to which the series converges
  - In fact, can rearrange the order to obtain any desired value

Advantages of absolute convergence

- Can continue to use distributive, commutative, and associate laws of sums

Examples

- $\sum_{k \geq 0} (-1)^k$

- $\sum_{k \geq 0} \frac{1}{k+1} + \sum_{k < 0} \frac{1}{k-1}$

- $\sum_{k \geq 0} \frac{-1}{2^k}$ [k is odd] + $\frac{1}{5^k}$ [k is even]

- $\sum_{k \geq 0} \frac{-1}{n}$ [k is odd] + $\frac{1}{5^k}$ [k is even]
Harmonic Series

- Lower and upper bounds on:
  \[ H_n = \sum_{k \leq n} \frac{1}{k} \]
- Put one term in group 1, two in group 2, four in group 3, etc.
- \[ 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + \ldots \]

- Each group has terms no bigger than first term and strictly bigger than first term of next group.
- \[ \frac{\lfloor \log_2 n \rfloor}{2} < H_n \leq \lfloor \log_2 n \rfloor + 1 \]
- Gives bound to \( H_n \) within a factor of 2

Better Bounds for Harmonic Series

Use integration of \( f(x) = \frac{1}{x} \)

- Lower bound:
  \[ \int_1^n \frac{1}{x} \, dx = \ln n \]
- Upper bound:
  \[ \frac{\lfloor \log_2 n \rfloor}{2} < H_n \leq \lfloor \log_2 n \rfloor + 1 \]

Bounds to \( H_n \) within at most 1