Sometimes Easier to Prove Stronger Result

In trying to devise a proof by mathematical induction, you may fail for two opposite reasons.

• You may fail because you try to prove too much: Your P(n) is too heavy a burden.
• Yet, you may also fail because you try to prove too little: Your P(n) is too weak a support.

In general, you have to balance the statement of your theorem so that the support is just enough for the burden.

—Polya

Example of Strengthening the Statement

**Given:**
- \( K_0 = 1 \)
- \( K_{n+1} = 1 + \min(2K_{\lfloor \frac{n}{2} \rfloor}, 3K_{\lfloor \frac{n}{3} \rfloor}) \)

**Show** \( K_n \geq n \)

- **Base case:**
- **Inductive step:**
  - Inductive hypothesis
  - To show:

Example

In a round-robin tournament, show that the n teams can be labeled \( T_1, T_2, \ldots, T_n \) such that \( T_i \) beats \( T_{i+1} \) for all \( 1 \leq i \leq n-1 \).
Induction and Algorithms

Algorithm correctness
- Precondition: predicates describing the initial state (involving the input variables)
- Postcondition: predicates describing the final state (involving the input and output variables)

Example:
- Product(int m, double r) // precondition: m a non-negative integer, r: a real
  // postcondition: returns m * r

Sort(int[] y)  // precondition: y is an array of integers
  // postcondition: y is a permutation of the original array such that
  y[i] ≤ y[i+1] for 0 ≤ i < y.length

Loop Invariant

Format:
- Loop precondition:
  Loop
  // Loop invariant:
  exit when ExitCondition
  Loop Contents
  End
- Loop postcondition:
  To Prove:
  - loop precondition → Loop invariant
  - Loop Invariant (at one iteration) ∧ Loop contents ∧ !ExitCondition → Loop Invariant (at next iteration)
  - ExitCondition is eventually true
  - Loop invariant ∧ ExitCondition → Loop postcondition

Infinite Sequences

An infinite sequence of real numbers:
- consists of terms: a0, a1, a2, ...
  - or, may start after 0:
    - a0, a1, ...
  - is a function f: N → R
    - or, may start after 0:
      - f: (N + n0) → R
- are normally written with subscripts rather than functional notation
  - can be a tail of another sequence:
    - a2, a3, a4, ... is a tail of a1, a2, a3, ...

Examples:
- 1/n for n ≥ 1
  - What is the function?
  - Domain?
- 2n+1 for n ≥ 3
  - What is the function?
  - Domain?
- 1/5, 1/6, 1/7, 1/8, ...
  - What is the function?
  - Domain?
- 1/n
  - What is the function?
  - Domain?
Limits

Given an infinite sequence \( a_n \):
- \( a_0, a_1, a_2, \ldots, a_n, \ldots \)
- \( A \) is the limit of \( a_n \) (as \( n \) approaches infinity), that is, \( \lim_{n \to \infty} a_n = A \)
- if, for all epsilon, there’s a tail of the sequence \( a_n \) whose values are all less than epsilon away from \( A \).

\[
\forall \epsilon > 0, \exists N\epsilon : \forall n \geq N\epsilon |A - a_n| \leq \epsilon
\]

If such a limit exists, we say that \( a_n \) \textit{converges} to \( A \).
If not, we say that \( a_n \) \textit{diverges}.
- Could diverge to \( \infty \)
- Could diverge to \(-\infty\)
- Could diverge to neither

Examples
- \( 0, 1, 2, 3, \ldots \)
- \( 1, 1, 1, 1, 1, 1, 1, \ldots \)
- \( 3, 3.1, 3.14, 3.141, 3.1415, \ldots, \pi \) to a million places, \( 3.14, 3.14, \ldots \)
- \( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots \)
- \( 1, -1, 10, -10, 100, -100, 1000, -1000, \ldots \)
- \( -10, -20, -30, -40, \ldots \)
- \( -1^n/n \)
- \( -1^k \)
- \( \sin(x) \)

Manipulating Limits

If \( a_n \) and \( b_n \) are sequences with limits \( A \) and \( B \), then:
- \( \text{limit of } a_n b_n = AB \)
- \( \text{limit of } Ca_n + Db_n = CA + DB \) \textit{linear combination}
- \( \text{limit of } a_n/b_n = A/B \) \textit{as long as } \( b_n \neq 0 \) for large enough \( n \)

Example
- \( \text{limit of } (3n+5)/(4n+3) \)
- \( \text{limit of } (\log n)/n \)
- \( \text{limit of } n/(n+1) \)
- \( \text{limit of } (\sin n)/n \)

L’Hospital’s rule: if limit of \( f(x) \) and \( g(x) \) are both 0 or both \( \infty \), then limit of \( f(x)/g(x) = \text{limit of } f'(x)/g'(x) \) (if such a limit exists)

Boundedness

A sequence \( a_n \) is \textit{bounded} if there is a positive number \( B \) such that for all elements, \( a_n \) of \( a_n \): \( |a_n| < B \).

Examples
- \( f(x) = \sin(x) \)
- \( 1, 1/2, 1/4, 1/8, \ldots \)

Convergent sequences are always bounded
Monotonicity

A sequence $a_n$ is
- **strictly increasing** if each term is bigger than the previous one
- **strictly decreasing** if each term is smaller than the previous one
- **nondecreasing** (or weakly increasing) if each term is no smaller than the previous one
- **nonincreasing** (or weakly decreasing) if each term is no bigger than the previous one
- **monotonic** (or monotone) if it is either nondecreasing or nonincreasing

A sequence is **eventually […]** if some tail of the sequence is […]

**Examples**
- $1, 3, 5, …$
- $x^2-8x+1$:

Bounded and Monotone

If a sequence is bounded and eventually monotone, then it converges.

**Example:**
- limit of $2 + 1/n$
- limit of $2 - 1/n$

Sandwich theorem:
- If $a_n$ has a limit of $A$ and $c_n$ has the same limit, and each term of $b_n$ has the property that:
  - $b_n \leq a_n \leq c_n$
- Then, the limit of $b_n$ is $A$

**Example:**
- limit of $2 + (-1)^n/n$

Sequence Cagematch

Given two sequences $a_n$ and $b_n$, which grows faster?

- Polynomial of degree $n$ versus polynomial of degree $n-1$
- Exponential versus polynomial?
- Polynomial versus logarithmic?

Identifying a Sequence

The Online Encyclopedia of Integer Sequences

|Enter part of a sequence:| 0 1 3 6 10 15 21 |

Get back a list of known related sequences

- **ID Number:** A000217 (Formerly M2535 and N1002)
- **Name:** Triangular numbers: $a(n) = C(n+1,2) = n(n+1)/2 = 0+1+2+...+n$.
- **Comments:**
  - Number of edges in complete graph of order $n$, $K_n$. Number of legal ways to insert a pair of parentheses in a string of $n$ letters. E.g. there are 6 ways for three letters: (ab)c, a(bc), abc, a(b)c, a(b)c, ab(c).
- **Proof:** There are $C(n+2,2)$ ways to choose where the parentheses might go, but $n+1$ of them are illegal because the parentheses are adjacent. Cf. A002415
- For $n >= 1$, $a(n) = n(n+1)/2$ is also the genus of a nonsingular curve of degree $n+2$ like the Fermat curve $x^{n+2} + y^{n+2} = 1$ - Ahmed Fares (ahmedfares(AT)my_deja.com), Feb 21 2001
- $a(n)$ is the number of ways in which $n+2$ can be written as a sum of three positive integers if representations differing in the order of the terms are considered to be different. In other words $a(n)$ is the number of positive integral solutions of the equation $x + y + z = n + 2$. - Amarnath Murthy (amarnath_murthy(AT)yahoo.com), Apr 22 2001