Powers of Permutations

To calculate the power of a function $f$, calculate the power of each cycle:

- Calculate final result for each element (using $i \mod \text{cycle-length}$)
- 1-cycles stay 1-cycles
- For example, if $f = (1, 2, 4), (3, 8, 7, 10), (5), (6), (9)$
  - $f^0 = (1, 2, 4), (3, 7, 6, 10), (5), (6), (9)$
- If all cycle lengths divide $i$ evenly, $f^i$ is the identity permutation
- If $f$ is a permutation of $A$, then $f^{|A|}$ is the identity permutation

Functions

Given $f : A \rightarrow B$

- **Inverse Image** of $f$:
  
  $$f^{-1}(b) = \{ a : a \in A \text{ and } f(a) = b \}$$

- **Coimage** of $f$: partition of $A$ where all elements in a block map to the same element of $B$
  
  $$\text{Coimage}(f) = \{ f^{-1}(b) : b \in \text{Image}(f) \}$$

Partition

Given a set $S = \{a_1, a_2, \ldots, a_n\}$:

- A partition $P$ is a set of subsets of $S$ such that:
  - the elements of $P$ cover $S$
  - The elements of $P$ are pairwise disjoint
  - No element of $P$ is empty
  - The elements of $P$ are called the **blocks** of the partition
Number of Partitions

Given \( S = \{a_1, a_2, \ldots, a_n\} \):
- How many partitions of \( S \) are there?
- How many partitions of size \( k \) are there?
  - If \( a_n \) is in a new block:
    - If \( a_n \) is in an existing block
  - \( S(n, k) \) is the \textit{Stirling number of the second kind} of \( \binom{n}{k} \)
  - \( S(n, k) = \)

Calculating \( S(n, k) \)

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Number of functions

Given sets \( A \) and \( B \), how many functions are there of the form \( f: A \rightarrow B \) where \( |\text{Image}(f)| = k \)
- How many partitions of \( A \) are there of size \( k \)?
- How many images are there of size \( k \)?
- How many ways to map the blocks of \( \text{Coimage}(f) \) to \( \text{Image}(f) \)?

Summing a sequence

Specific sum
- \( 1 + 2 + 3 + \ldots + n-1 + n \)

General summation of a sequence of terms:
- \( a_1 + a_2 + \ldots + a_n \)
Sigma notation

Ellipsis (…) notation is vague and wordy
Sigma notation is more compact:
\[ \sum_{k=1}^{n} k \]

Parts of the notation
- Summand
- Index variable
- Lower limit
- Upper limit

Sigma notation inline: \( \sum_{k=1}^{n} k \)

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Sigma notation inline: \( \sum_{k=1}^{n} k \)

Changing the index variable

Can always change from one index variable to another:
\[ \sum_{1 \leq k \leq n} a_k = \]

What if we want to switch from \( k \) to \( k+1 \)?
- Compare:
  \[ \sum_{1 \leq k \leq n} a_k = \]
- To:
  \[ \sum_{k=1}^{n} a_k = \]

Generalized Sigma notation

Specify a condition that the index variable must satisfy:
\[ \sum_{1 \leq k \leq n} k \]

- Compare:
  \[ \sum_{1 \leq k \leq 100} k^2 \]
- To:
  \[ \sum_{k=0}^{49} (2k+1)^2 \]

Zero terms are OK

Which is better?
\[ \sum_{k=2}^{n-1} k(k-1)(n-k) \]

or:
\[ \sum_{k=0}^{n} k(k-1)(n-k) \]
Using Iverson Notation

Iverson notation. True-or-false statement enclosed in brackets. Value is 0 or 1

- \([p \text{ odd}] = \sum_{k \in K \text{ odd}} k^2\]

Example:

\[\sum_{1 \leq k \leq 100} k^2 = \sum_{k} k^2[1 \leq k \leq 100][k \text{ odd}]\]

0 in Iverson notation is very strongly zero

\[\sum_{k} \frac{1}{k}[k > 0]\]

Products

Pi notation is similar to Sigma notation

\[\prod_{k=1}^{n} k = 1 \times 2 \times 3 \times \cdots (n-1) \times n\]

Working with summations

Distributive law

\[\sum_{k \in K} c a_k = c \sum_{k \in K} a_k\]

Associative law

\[\sum_{k \in K} a_k + b_k = \sum_{k \in K} a_k + \sum_{k \in K} b_k\]

Commutative law

\[\sum_{k \in K} a_k = \sum_{p(k) \in K} a_k\]

Closed forms for summations

Closed form is a formula for a summation with the summation removed.

\[\sum_{0 \leq k \leq n} k = \frac{n(n + 1)}{2}\]
Example

\[ \sum_{0 \leq k \leq n} (a + bk) = \]

Perturbation method

Given:

\[ S_n = \sum_{0 \leq k \leq n} a_k. \]

Rewrite \( S_{n+1} \) by splitting off first and last term:

\[ S_n + a_{n+1} = a_0 + \sum_{1 \leq k \leq n+1} a_k \]
\[ = a_0 + \sum_{1 \leq k+1 \leq n+1} a_k \]
\[ = a_0 + \sum_{0 \leq k \leq n} a_k \]

Then, work on last sum and express in terms of \( S_n \). Finally, solve for \( S_n \).

Perturbation method example

\[ S_n = \sum_{0 \leq k \leq n} x^k \]

Standard closed forms

Arithmetic series

\[ \sum_{k=0}^{n} k = \frac{1}{2} n(n + 1) \]

Sums of squares and cubes

\[ \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]
\[ \sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \]
Standard closed forms

Geometric series ($x \neq 1$)

\[
\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}
\]

Infinite Geometric series ($|x| < 1$)

\[
\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}
\]

Harmonic series

\[
H_n = \sum_{k=0}^{n} \frac{1}{k} \approx \ln n
\]

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Application of harmonic series

Given a stack of 52 playing cards (of length 2 units). Can you stack them overlapping so the top card completely overhangs the table?