Sets

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Counting

Given the set \( S = \{x, y, z\} \)

- How many ways to list (without duplicates) all elements of \( S \)?
  - Permutations of \( S \)
  - How many ways to list (with duplicates) all elements of \( S \)?
  - How many ways to construct a \( k \)-list of distinct elements from \( S \)?
  - How many ways to construct a 2-set of (distinct) elements?
  - How many ways to construct a 2-multiset of elements?

Permutations and Combinations

Permutation of \( n \) objects: ordering of all \( n \) objects

- Order matters
  - \( P(n, 2) = n(n-1) \)
  - \( P(n, 3) = n(n-1)(n-2) \)
  - \( P(n, r) = n(n-1)(n-2)...(n-(r-1)) \)
  - \( P(n, n) = n(n-1)...(2)(1) = n! \)
  - \( P(n, r) = n(n-1)(n-2)...(n-(r-1)) = n!/(n-r)! = \binom{n}{r} \)

- How many ways are there to rank \( n \) candidates for the job of chief wizard?
  - If rankings are chosen at random, what is the probability that the fifth candidate is in second place?

Example Problem

- How many ways are there to rank \( n \) candidates for the job of chief wizard?
  - If rankings are chosen at random, what is the probability that the fifth candidate is in second place?
Example Problem

If a 5-card hand is chosen at random, what is the probability of a flush (all 5 cards of the same suit)?

- How many 5-card hands are there?
- How many 5-card hands are flushes?

Binomial Theorem

Let's look at 

\[(x+y)^n\]

- What is the coefficient of \(x^{n-k}y^k\) in \((x+y)^n\)?

Binomial Theorem:

\[(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + \binom{n}{n}y^n\]

Binomial Identities

- Symmetry Identity

\[\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}\]

- Fundamental Identity

\[\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}\]

- Algebraic Proof

- Proof by Pascal's triangle
- Proof by combinatorial argument
Pascal's Triangle

Number of paths equals \( C(n, k) \)

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<th>2</th>
<th>3</th>
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Functions

Give two sets \( A \) and \( B \), a function is a rule that tells how to find a unique \( b \in B \) for each \( a \in A \).

- It is shown as \( f : A \rightarrow B \)
- \( A \) is the domain
- \( B \) is the range (or codomain)
- The set of all functions from \( A \) to \( B \) is written \( B^A \)
- Given a set \( S \subseteq A \), \( f(S) = \{ f(x) : x \in S \} \)
- \( f(S) \) is called image

Ways of showing the rule (given domain and range: \( A = \{ \text{red, green, blue} \} \), \( B = \{0, 28\} \)):

- One-line notation
  - If \( A \) is ordered, give the corresponding values of \( B \):
    - Ordering of \( A = \{ \text{red, green, blue} \} \), \( f = \{5, 8, 27\} \)

- Two-line notation
  - Give ordering of \( A \) and values of \( B \) at the same time:
    - \( f = (\text{red, green, blue}) \):
      - \( 5, 8, 27 \)

- Set notation: \( \{(\text{red, 5}), (\text{blue, 27}), (\text{green, 8})\} \)

Types of Functions

Surjection (onto): Every value in the codomain is taken on at least once

- \( \forall b \in B, \exists a \in A : f(a) = b \)

Injection (one-to-one): Every value in the codomain is taken on at most once.

- \( \forall a \in A, b \in B, f(a) = f(b) \Rightarrow a = b \)

Bijection (one-to-one and onto): Every value in the codomain is taken on exactly once.

- \( \forall b \in B, \exists ! a \in A : f(a) = b \)

Example

Let

- \( S \) be the set of students attending UCSD
- \( I \) be the set of student ID numbers for those students
- \( D \) be the set of dates (MM/DD/YYYY) for the past 100 years
- \( G \) be the set of possible grade point averages between 2.0 and 3.5 (rounded to the nearest tenth)

Is the following an injection, bijection, or surjection, or none?

- The domain is \( S \), the codomain is \( I \) and the function maps each student to his/her ID number
- The domain is \( S \), the codomain is \( D \) and the function maps each student to his or her birthday
- The domain is \( D \), the codomain is \( I \) and the function maps each date to the ID number of a student born on that date. If more than one, the lexicographically least ID number is chosen.
- The domain is \( S \), the codomain is \( G \) and the function maps each student to his or her GPA rounded to the nearest tenth.
- The domain is \( G \), the codomain is \( I \) and the function maps each GPA to the ID number of a student with that GPA. If there is more than one, the lexicographically least ID number is chosen.
Example

Bijective, injective, surjective, or neither?
- $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x$
- $f: \mathbb{R}^+ \to \mathbb{R}^+ f(x) = x^2$
- $f: \mathbb{R} \to \mathbb{R} f(x) = x^2$
- $f: \mathbb{R}^+ \to \mathbb{R}^+ f(x) = \sqrt{x}$
- $f: \mathbb{R} \to \mathbb{R} f(x) = (x-1)x(x+1)$
- $f: \mathbb{R} \to \mathbb{R} f(x) = e^x$
- $f: \mathbb{R} \to \mathbb{R} f(x) = e^x$

Bijections

Given two sets $A$ and $B$
- $|A| = |B|$ if there exists a bijection $f: A \to B$

Inverse Functions

If $f: A \to B$ is a bijection, then $f^{-1}$ is the **inverse** of $f$. $f^{-1}: B \to A$
- if $f(a) = b$, then $f^{-1}(b) = a$

Numbers of Functions

How many functions of the form $f: A \to B$?
- Equivalently, what is $|B^A|$?
- How many choices for first element of $A$?
- How many choices for second element of $A$?
- …
- How many choices for last element of $A$?