Common Knowledge Problem

There are $n$ people, each wearing a red or black hat.
Each can see the others' hats, but not own hat.
Hint told publicly: “At least one of you has a black hat”
Each is then asked: “Is your hat black?”

Questions for You

How do each figure out they have a black hat?
Why is the hint necessary?
• After all, each can see someone with a black hat
• Does it have to be public?
What does this have to do with distributed systems???
What If Only One Black Hat?

A sees 2 red hats, knows there is 1 black hat (hint)
A can conclude “I must have a black hat”
So, after first question, A answers yes

What If Two Black Hats?

A and B see 1 red hat and 1 black hat
No one can answer first question
A says: “If I had red hat, B would have answered”
So, upon second question, A (and B) can answer yes

What If Three Black Hats?

Each sees 2 black hats
No one can answer first or second question
Each say: If I had red hat, the others would have answered before, therefore I must have black hat

Why is the Hint Necessary?

The hint allows each to tell the difference between
• All red hats vs. one black hat
It provides information about what the others know
1st Time Asked: What Does A Know?

- That B and C have red hats (by observation)
- That there is at least one black hat (the hint)
  - Since everyone knows the hint
- Deduction: A deduces it must have black hat

2nd Time Asked: What Does A Know?

- That B has black, C has red hat (by observation)
- That there is at least one black hat (the hint)
- That B and C know the hint
  - Since everyone knows that everyone knows the hint
- Deduction: A deduces it must have black hat

3rd Time Asked: What Does A Know?

- That B and C have black hats (by observation)
- That there is at least one black hat (the hint)
- That B and C know the hint
- That B and C know that A knows the hint
  - Since everyone knows that everyone knows that everyone knows the hint
- Deduction: A deduces it must have black hat

Common Knowledge

Knowledge of x
- \( K(x) = \) Everyone knows x
- \( K^2(x) = \) Everyone knows that everyone knows x
- \( K^3(x) = \) Everyone knows that everyone knows that everyone knows x

Common knowledge of x
- \( K(x) \) and \( K^2(x) \) and ... \( K^n(x) \) as \( n \to \infty \)

By announcing hint, it becomes common knowledge
- Announcement must be public!