Image Formation, Cameras, Radiometry

Computer Vision I
CSE 252A
Lecture 4-Part I

Announcements
• Read Chapters 1 & 2 of Forsyth & Ponce
• Discussion board created by Eric Weigle

Last lecture in a nutshell

Pinhole Camera: Perspective projection
• Abstract camera model - box with a small hole in it

Geometric properties of projection
• Points go to points
• Lines go to lines
• Planes go to whole image or half-plane
• Polygons go to polygons
• Angles & distances not preserved
• Degenerate cases:
  – line through focal point yields point
  – plane through focal point yields line

Parallel lines meet in the image
• vanishing point
Projective Geometry

- Axioms of Projective Plane
  1. Every two distinct points define a line
  2. Every two distinct lines define a point (intersect at a point)
  3. There exists three points, A, B, C such that C does not lie on the line defined by A and B.

- Different than Euclidean (affine) geometry
- Projective plane is “bigger” than affine plane – includes “line at infinity”

Homogenous Coordinates & the camera matrix

- Homogenous Coordinates
  - Euclidean -> Homogenous: (x, y) -> k (x, y, 1)
  - Homogenous -> Euclidean: (x, y, z) -> (x/z, y/z)

- Homogenous translation
  - Points map to points, lines map to lines

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix} =
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & x_1 \\
  a_{21} & a_{22} & a_{23} & x_2 \\
  a_{31} & a_{32} & a_{33} & x_3
\end{bmatrix}
\]

- Perspective
  - Assume that \( f = 1 \), and perform a Taylor series expansion about \((x_0, y_0, z_0)\)

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}
\approx
\begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
\]

- Dropping higher order terms and regrouping.

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}
= \begin{bmatrix}
  1 & 1/z_0 & 0 & -x_0/z_0^2 \\
  0 & 1/z_0 & 0 & -y_0/z_0^2 \\
  0 & 0 & 1 & z_0
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = Ap + b
\]

Orthographic projection

- \( x' = x \)
- \( y' = y \)

Homogenous Coordonates and Camera matrix

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
\]

What if camera coordinate system differs from object coordinate system

- Translation & Rotation between Coordinate Systems
Rotation Matrix

\[
\mathbf{R} = \begin{bmatrix}
\mathbf{i}_k & \mathbf{j}_k & \mathbf{k}_k \\
\mathbf{i}_a & \mathbf{j}_a & \mathbf{k}_a \\
\mathbf{i}_b & \mathbf{j}_b & \mathbf{k}_b \\
\end{bmatrix} = \begin{bmatrix}
\mathbf{i}_a & \mathbf{j}_a & \mathbf{k}_a \\
\mathbf{i}_b & \mathbf{j}_b & \mathbf{k}_b \\
\mathbf{i}_c & \mathbf{j}_c & \mathbf{k}_c \\
\end{bmatrix}
\]

Coordinate Changes: Rigid Transformations

\[
B \mathbf{p} = A \mathbf{R} A \mathbf{p} + B \mathbf{O}_A
\]

Some points about SO(n)

- \( \text{SO}(n) = \{ R \in \mathbb{R}^{n \times n} : R^T R = I, \det(R) = 1 \} \)
  - \( \text{SO}(2) \): rotation matrices in plane \( \mathbb{R}^2 \)
  - \( \text{SO}(3) \): rotation matrices in 3-space \( \mathbb{R}^3 \)
- Forms a Group under matrix product operation:
  - Identity
  - Inverse
  - Associative
  - Closed (finite intersection of closed sets)
- Bounded \( R_{ij} \in [-1, 1] \)
- Does not form a vector space.
- Manifold of dimension \( n(n-1)/2 \)
  - \( \dim(\text{SO}(2)) = 1 \)
  - \( \dim(\text{SO}(3)) = 3 \)

SO(3)

- Parameterizations of SO(3)
- 3-D manifold, so between 3 parameters and \( 2n+1 \) parameters (Whitney’s Embedding Thm.)
  - Roll-Pitch-Yaw
  - Euler Angles
  - Axis Angle (Rodrigues formula)
  - Cayley’s formula
  - Matrix Exponential
  - Quaternions (four parameters + one constraint)

Rigid Transformations as Mappings: Rotation about the \( k \) Axis

\[
\mathbf{F}' = \mathbf{F}^\mathbf{R}, \quad \text{where} \quad \mathbf{R} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = \text{rot}(k, \theta)
\]

Rotation: Homogenous Coordinates

- About z axis

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
Rotation

- About x axis:

\[
\begin{bmatrix}
  x' \\
y' \\
z'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & \cos \theta & -\sin \theta & 0 \\
  0 & \sin \theta & \cos \theta & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

- About y axis:

\[
\begin{bmatrix}
  x' \\
y' \\
z'
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & 0 & \sin \theta & 0 \\
  0 & 1 & 0 & 0 \\
  -\sin \theta & 0 & \cos \theta & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

\[
\text{Roll-Pitch-Yaw}
\]

\[
R = \text{rot}(i, \alpha)\text{rot}(j, \beta)\text{rot}(k, \varphi)
\]

Euler Angles

\[
R = \text{rot}(k^\prime, \alpha)\text{rot}(j^\prime, \beta)\text{rot}(k, \varphi)
\]

Quaternions

\[
q = (a, \alpha)
\]

\[
\text{q is a quaternion (generalization of imaginary numbers)}
\]

\[
a \in \mathbb{R} \text{ is its real part}
\]

\[
\alpha \in \mathbb{R}^3 \text{ is its imaginary part.}
\]

Operations on quaternions:

- Sum of quaternions: \((a, \alpha) + (b, \beta) = ((a+b), \lambda_\alpha, \lambda_\beta, \alpha \times \beta)\)
- Multiplication by a scalar: \(\lambda (a, \alpha) = (\lambda a, \lambda \alpha)\)
- Quaternion product:

\[
(a, \alpha)(b, \beta) = ((a b - \alpha \cdot \beta), (a b + \alpha \times \beta))
\]

- Conjugate: \(q = (a, \alpha)\) \(q^* = (a, -\alpha)\)
- Norm: \(|q|^2 = q q^* = q^* q = a^2 + |\alpha|^2\)

Unit Quaternions and Rotations

- Let \(R\) denote the rotation of angle \(\theta\) about the unit vector \(u\).
- Define unit quaternion \(q = (\cos \theta/2, \sin \theta/2 u)\).
- Note \(|q| = 1\) (i.e., \(q\) lies on unit sphere for any \(u\) and \(\theta\)).
- Then for any vector \(a\), \(R a = \text{imaginary}(q a^* q^*)\)
- \(q\) and \(-q\) define the same rotation matrix.

If \(q = (a, (b, c, d))^T\) is a unit quaternion, the corresponding rotation matrix is:

\[
R = \begin{bmatrix}
  a^2 + b^2 - c^2 - d^2 & 2(bc + ad) & 2(bd - ac) & 0 \\
  2(bc - ad) & a^2 - b^2 + c^2 - d^2 & 2(cd + ab) & 0 \\
  2(bd + ac) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

Block Matrix Multiplication

\[
A = \begin{bmatrix}
  A_1 & A_2 \\
  A_3 & A_4
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
  B_1 & B_2 \\
  B_3 & B_4
\end{bmatrix}
\]

What is \(AB\) ?

\[
AB = \begin{bmatrix}
  A_1 B_1 + A_2 B_3 & A_1 B_2 + A_2 B_4 \\
  A_3 B_1 + A_4 B_3 & A_3 B_2 + A_4 B_4
\end{bmatrix}
\]

Homogeneous Representation of Rigid Transformations

\[
^p R = \begin{bmatrix}
  ^p R & ^p O \vert ^p 1
\end{bmatrix}
\]

\[
^p P = \begin{bmatrix}
  ^p R & ^p P \vert ^p 1
\end{bmatrix}
\]

\[
= ^x R (\begin{bmatrix}
  ^x P \\
  1
\end{bmatrix})
\]
Camera parameters

- Issue
  - camera may not be at the origin, looking down the z-axis
    - extrinsic parameters
  - one unit in camera coordinates may not be the same as one unit in world coordinates
    - intrinsic parameters - focal length, principal point, aspect ratio, angle between axes, etc.

\[
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix} = \text{Transformation representing intrinsic parameters}
\begin{bmatrix}
\lambda \\
\gamma
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z \\
T
\end{bmatrix} = \text{Transformation representing extrinsic parameters}
\]

Camera Calibration

Given n points \( P_1, \ldots, P_n \) with known positions and their images \( \tilde{P}_1, \ldots, \tilde{P}_n \), estimate intrinsic and extrinsic camera parameters

- See Text book for how to do it.

Beyond the pinhole Camera

Getting more light – Bigger Aperture

Pinhole Camera Images with Variable Aperture

The reason for lenses
Thin Lens

- Rotationally symmetric about optical axis.
- Spherical interfaces.

Thin Lens: Center

- All rays that enter lens along line pointing at $O$ emerge in same direction.

Thin Lens: Focus

Parallel lines pass through the focus, $F$

Thin Lens: Image of Point

All rays passing through lens and starting at $P$ converge upon $P'$

Thin Lens: Image Plane

\[ \frac{1}{z'} - \frac{1}{z} = \frac{1}{f} \]

A price: Whereas the image of $P$ is in focus, the image of $Q$ isn’t.
Deviations from the lens model

Deviations from this ideal are **aberrations**

Two types

1. geometrical
   - spherical aberration
   - astigmatism
   - distortion
   - coma

2. chromatic
   Aberrations are reduced by combining lenses

**Spherical aberration**

rays parallel to the axis do not converge

outer portions of the lens yield smaller focal lengths

**Distortion**

magnification/focal length different for different angles of inclination

Can be corrected! (if parameters are know)

**Chromatic aberration**

(great for prisms, bad for lenses)
Chromatic aberration:

- Rays of different wavelengths focus in different planes.
- This cannot be removed completely.
- Sometimes, achromatization is achieved for more than 2 wavelengths.

Vignetting:

- The image is blurr and appears colored at the fringe.