# Quiz 3 Solutions — CSE 105, Winter '03

## Problem 1 [8 pt].

Consider the TM in the picture below and answer the following questions:

- What is the initial configuration on input $aababb$?

  ANSWER: $(\epsilon, q_s, aababb)$.

- What is the length of the computation on input $aababb$?

  ANSWER:7 steps

- What is the final configuration on input $aababb$?

  ANSWER: $(aabbab, q_a, b)$

- Does $M$ accepts $aababb$?

  ANSWER: YES

- Give an example of string that is rejected by $M$

  ANSWER: $\epsilon$

- Give an example of string that is neither accepted nor rejected by $M$

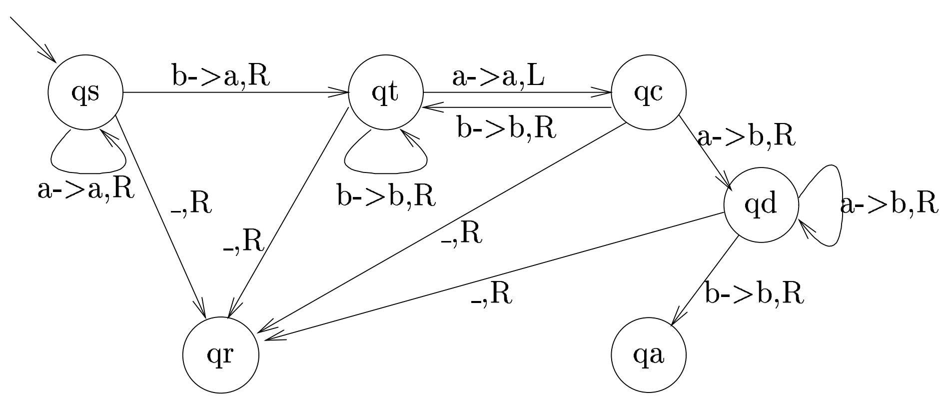
  ANSWER: $bba$

- Is $M$ a decider?

  ANSWER: NO (becuase it does not terminate on input $bba$)

- Is the language accepted by $M$ regular?

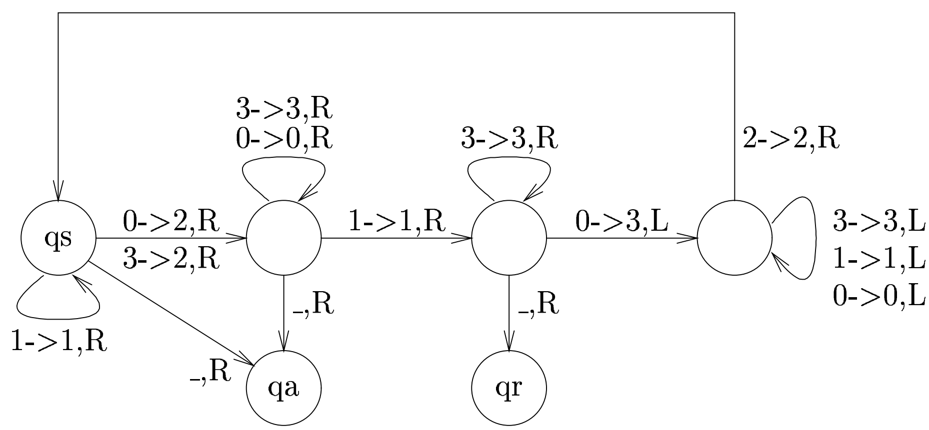
  ANSWER: YES (it is the language $L(a^*baa^*b(a+b)^*)$)



## Problem 2 [8 pt].

Give an implementation level description of a TM recognizing the set of all words $w \in \{0,1\}^*$ of the form $w = 0^{a_1}10^{a_2}1\ldots10^{a_n}$ such that $a_1 \le a_2 \le \ldots \le a_n$. (Remember, implementation level description means an informal description of a state transition diagram, i.e., although you do not have to draw the state transition diagram, your description should be detailed enough so that it can be translated into a diagram with little or no effort.) If you prefer, giving a state transition diagram is also an acceptable solution to this problem, but your solution should be clear and reasonably concise. (Notice: there are TMs recognizing this language with as low as 6 states, including $q_s, q_a$ and $q_r$.)

ANSWER: We give a state transition diagram (there are many other possible solutions):




## Problem 3 [8 pt].

Let $L$ be the language of all strings of the form $\langle R, w\rangle$ where $R$ is a regular expression over the binary alphabet $\Sigma = \{0,1\}$, and $w = w_1, \ldots, w_n$ is a comma-separated list of words such that for all $i = 1, \ldots, n$, $w_i \in L(R)$ if and only if $n$ is odd. (I.e., if $n$ is odd then all $w_i$ should belong to $L(R)$, while if $n$ is even then none of them should belong to $L(R)$.)

Prove that $L$ is decidable. (You can either build a decider from scratch, or reduce the problem to some other decidable problem studied in class.)

ANSWER: We reduce the language $L$ to the acceptance problem for regular expressions $A_{RE}$. Assume $D$ is a decider for $A_{RE}$. We define a decider for $L$ as follows:

1. On input $\langle R, w\rangle$ check that $R$ is a regular expression over the alphabet $\{0,1\}$ and $w$ is a string over $\{0, 1, '','' \}$. If not, reject immediately.

2. Build a regular expression $S$ for the complement of the language $L(R)$. (This can be done by first transform $R$ into an equivalent DFA, then changing the accepting and non-accepting states in the DFA, and transform the resulting DFA into an equivalent regular expression $S$ using the procedures studied in class.

3. Build regular expression

$$T = ((R, R,)^* R) \cup ((S, S,)^* S, S) \cup \epsilon.$$

4. Check if $\langle T, w \rangle \in A_{RE}$ using decider $D$. If $D$ accepts, then accept, otherwise reject.

# Problem 4 [8 pt].

Consider a variant of TM that at each step can either overwrite the current tape cell (without moving the tape head), or move the tape head (leaving the current tape cell unaltered). Formally, this simplified TM can be defined as a tuple $M = (Q, \Sigma, \Gamma, \delta, q_s, q_a, q_r)$ where $Q, \Sigma, \Gamma, q_s, q_a, q_r$ are defined as for the standard Turing Machine, and the transition function is of type $\delta\colon Q \times \Gamma \to Q \times (\Gamma \cup \{L, R\})$, where $L, R$ are two special symbols not belonging to $\Gamma$. Computation steps are defined in the obvious way: if TM is in state $q$, the tape head is positioned on a cell containing the symbol $a$, and $\delta(q, a) = (q', x)$, then TM changes internal state to $q'$, and either to overwrites the $a$ with $x$ (if $x \in \Gamma$), or moves the tape head in direction $x$ (if $x = L/R$).

Prove that this simplified model of computation is as powerful as an ordinary Turing Machine, giving a transformation from a generic TM $M = (Q, \Sigma, \Gamma, \delta, q_s, q_a, q_r)$ into a machine $M' = (Q', \Sigma, \Gamma, \delta', q'_s, q'_a, q'_r)$ satisfying the new definition. You should give first an informal (short) description of the transformation, followed by a formal definition of $M'$ in terms of $M$. ($M'$ should have the same input and tape alphabets as $M$. What you have to do is to define the new $Q'$, $\delta'$ and $q'_s, q'_a, q'_r$ in terms of $M$.)

ANSWER: The set of states is $Q' = Q \times \{L, R, S\}$. The start, accept and reject states are

$$\begin{aligned} q'_s &= (q_s, S) \\ q'_a &= (q_a, S) \\ q'_r &= (q_r, S) \end{aligned}$$

and the transition function is defined by

$$\begin{aligned} \delta'((q, S), a) &= ((q', D), b) \text{ if } \delta(q, s) = (q', b, D) \\ \delta'((q, L), a) &= ((q, S), L) \\ \delta'((q, R), a) &= ((q, S), R) \end{aligned}$$

for all $a, b \in \Gamma$, $q, q' \in Q$, $D \in \{L, R\}$.

# Problem 5 [8 pt].

Consider the language $L = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is not context free}\}$. Prove that $L$ is undecidable.

ANSWER: The proof is by reduction from $A_{TM}$. Assume $D$ is a decider for $L$. We use $D$ to decide $A_{TM}$ as follows:

1. On input $\langle M, w \rangle$, check that $M$ is a TM, and $w$ a valid string over the input alphabet of $M$. If not, reject immediately.

2. Build a new TM $M'(x)$ that works as follow:

   (a) If $x = a^n b^n c^n$ for some $n \geq 0$ then accept

   (b) Otherwise, run $M$ on input $w$.

   (c) If $M$ accepts $w$ then, $M'$ accepts $x$. (If $M$ rejects $w$ or loops, then $x$ is not accepted by $M'$.)

3. Notice: if $\langle M, w \rangle \in A_{TM}$ then $L(M') = \Sigma^*$ which is context free (in fact, it is even regular). Otherwise, $L(M') = \{a^n b^n c^n : n \geq 0\}$ which is not context free.

4. Run $D$ on input $\langle M' \rangle$. If $D$ accepts, then reject. If $D$ rejects, then accepts.