Discussion Section Notes
Pumping Lemma for Context Free Languages (PL4CFL)

Example of Problem and Solution

Problem 1: Consider \( L = \{ tt : t \in \{ a, b \}^* \} \). Prove it is not context free.

Solution:
No, it is not. We'll prove it by using the PL4CFL.

Assume by contradiction that \( L \) is context free (CF). Then, by PL4CFL, we know that there exists a pumping length \( p > 0 \) such that any word \( w \in L, |w| \geq p \), can be partitioned as \( uvxyz = w \) (with \( |vxy| \leq p \) and \( |uv| > 0 \)) in such a way that, for any \( i \geq 0 \), the word \( uv^i xy^i z \) also belongs to \( L \).

One (incorrect) possible choice is \( w = a^p b^p \). However, this string can be pumped. Check it.

On the other hand, consider the string \( w = a^p b a^p b^p \). Clearly \( |w| \geq p \). Moreover, any partition of \( w \) into \( uvxyz \) must fall into exactly one of the following cases:

1. \( v \) and \( y \) both contain only the same symbol, that is, either \( v = a^k \) and \( y = a^j \), or \( v = b^k \) and \( y = b^j \). By pumping down this word (which means \( i = 0 \)) into \( w' = uzx \) we get that \( w' = a^m b^n b^p \) where all except one exponent are equal to \( p \). The offending exponent is less than \( p \) since either \( k + j > 0 \). Therefore, string \( w' \) cannot belong to language \( L \). We get a contradiction for this case.

2. Either \( v \) and \( y \) contains two different symbols, that is, either \( v = a^k b^j \) and \( y = b^l \), or \( v = b^k a^j \) and \( y = a^l \). By pumping up this word (say \( i = 2 \)) into \( w' = uv^2 xy^2 z \) we get one of the two following cases:

   - \( v = a^k b^j \) and \( y = b^l \), and assume the substring \( vxy \) is in the first half of \( w \); the case \( vxy \) is in the second half is analogous. Then \( w' = a^{p-k} (a^k b^j)^2 b^{p-j-l} (b^l)^2 a^p b^p = a^{p-k} a^k b^j a^k b^j b^p a^p b^p = a^p b^j a^k b^{p+i} a^p b^p \). There is no way to split \( w' \) into two equal strings \( tt \) in order to satisfy the restriction \( w' = tt \) for some \( t \in \{ a, b \}^* \). This follows from the fact that, if \( w' \) were in \( L \), the string \( t \) should start with an \( a \), and the two possible ways to break \( w' \) into \( tt \) such that \( t \) starts with an \( a \) contain a different combination of \( a \)'s and \( b \)'s.

   - \( v = b^k a^j \) and \( y = a^l \). Then \( w' = a^{p-2} b^{p-k} (b^k a^j)^2 b^{p-j-l} (a^l)^2 a^p b^p = a^p b^j a^k b^k a^{p+i+l} b^p \). There is no way to split \( w' \) into two equal strings \( tt \) in order to satisfy the restriction \( w' = tt \) for some \( t \in \{ a, b \}^* \). The same argument as the one given in the previous case applies here.
The two sub-cases imply that string $w'$ cannot belong to language $L$. We get a contradiction for this case.

3. $v = a^k$ and $y = b^j$, or $v = b^k$ and $y = a^j$. By pumping down ($i = 0$) $w$ onto $w' = uxz$ we get that $w' = uxz = a^{p-k}b^p-jap^p$ (if $v = a^k$ and $y = b^j$, and the substring $vxy$ is in the first half of $w$) Since $k + j > 0$ we know that either $p - k$ or $p - j$ is less than $p$. Similar results are obtained when considering the case $v = b^k$ and $y = a^j$ and/or the substring $vxy$ is in the second half of $w$. Therefore, string $w'$ cannot belong to language $L$. We get a contradiction for this case.

Since for any possible partition of $w$ into $uvxyz$ we obtain a contradiction, we have that $L$ is not context free.