Discussion Section Notes
Pumping Lemma for Context Free Languages (PL4CFL)

Example of Problem and Solution

Problem 1: Consider $L = \{\, tt \,:\, t \in \{\, a, b \,\}^* \,\}$. Prove it is not context free.

Solution:

No, it is not. We'll prove it by using the PL4CFL.

Assume by contradiction that L is context free (CF). Then, by PL4CFL, we know that there exists a pumping length $p > 0$ such that any word $w \in L$, $|w| \geq p$, can be partitioned as $uvxyz = w$ (with $|vxy| \leq p$ and $|uy| > 0$) in such a way that, for any $i \geq 0$, the word uv^ixy^iz also belongs to L.

One (incorrect) possible choice is $w = a^pa^p$. However, this string can be pumped. Check it.

On the other hand, consider the string $w = a^pb^pa^pb^p$. Clearly $|w| \geq p$. Moreover, any partition of w into $uvxyz$ must fall into exactly one of the following cases:

1. v and y both contain only the same symbol, that is, either $v = a^k$ and $y = a^j$, or $v = b^k$ and $y = b^j$. By pumping down this word (which means $i = 0$) into $w' = uxz$ we get that $w' = a^\alpha b^\beta a^\gamma b^\delta$ where all except one exponent are equal to p. The offending exponent is less than p since either $k + j > 0$. Therefore, string w' cannot belong to language L. We get a contradiction for this case.

2. Either v and y contains two different symbols, that is, either $v = a^kb^j$ and $y = b^l$, or $v = b^ka^j$ and $y = a^l$. By pumping up this word (say $i = 2$) into $w' = uv^2xy^2z$ we get one of the two following cases:

   - $v = a^kb^j$ and $y = b^l$, and assume the substring vxy is in the first half of w; the case vxy is in the second half is analogous. Then $w' = a^{p-k}(a^kb^j)^2b^{p-j-l}(b^l)^2a^pb^p = a^{p-k}a^kb^ja^kb^jb^{p-j-l}b^lb^la^pb^p = a^pb^ja^kb^{p+l}a^pb^p$. There is no way to split w' into two equal strings tt in order to satisfy the restriction $w' = tt$ for some $t \in \{a, b\}^*$. This follows from the fact that, if w' were in L, the string t should start with an a, and the two possible ways to break w' into tt such that t starts with an a contain a different combination of $a's$ and $b's$.
   - $v = b^ka^j$ and $y = a^l$. Then $w' = a^pb^{p-k}(b^ka^j)^2a^{p-j-l}(a^l)^2b^p = a^pb^pa^jb^ka^{p+l}b^p$. There is no way to split w' into two equal strings tt in order to satisfy the restriction $w' = tt$ for some $t \in \{\, a, b \,\}^*$. The same argument as the one given in the previous case applies here.

The two sub-cases imply that string w' cannot belong to language L. We get a contradiction for this case.

3. $v = a^k$ and $y = b^j$, or $v = b^k$ and $y = a^j$. By pumping down ($i = 0$) w onto $w' = uxz$ we get that $w' = uxz = a^{p-k}b^{p-j}a^p b^p$ (if $v = a^k$ and $y = b^j$, and the substring vxy is in the first half of w) Since $k + j > 0$ we know that either $p - k$ or $p - j$ is less than p. Similar results are obtained when considering the case $v = b^k$ and $y = a^j$ and/or the substring vxy is in the second half of w. Therefore, string w' cannot belong to language L. We get a contradiction for this case.

Since for any possible partition of w into $uvxyz$ we obtain a contradiction, we have that L is not context free.