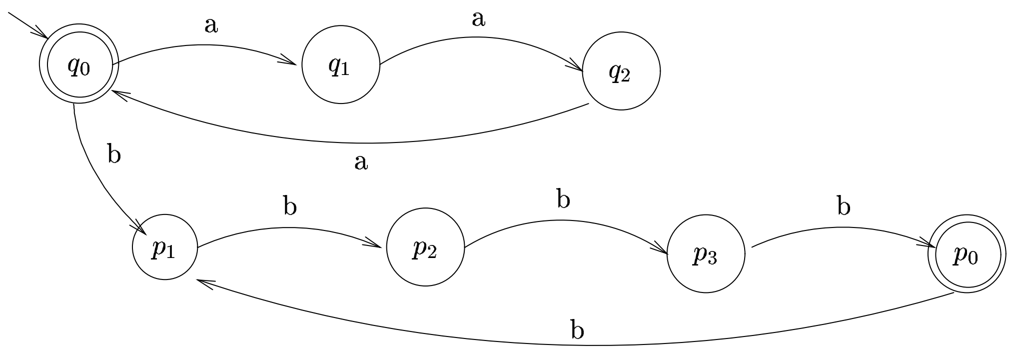
Discussion Section Notes – Solutions of Suggested Problems

Problem 1: Consider $L = \{ a^{3n}b^{4m} \ : \ n, m \geq 0 \}$. Is it regular or not? Prove it.

Solution: Yes, it is regular. The following is the DFA that recognizes it:



Although the language L seems very similar to some nonregular languages like 0^n1^n, there is one important difference: in this case there is no unbounded amount of information to remember. "Wait a second" –you may say– "Doesn't the automaton have to remember the number of $a's$ in order to check if it is multiple of 3?" Not really. The automaton only needs to remember whether the number of initial $a's$ is a multiple of 3. This task can be done by creating one state for the case "number of $a's$ is exactly multiple of 3" (q_0), another state for "number of $a's$ is one more than a multiple of 3" (q_1), and another one for "number of $a's$ is two more than a multiple of 3" (q_2). After that, it is easy. Only on state q_1 the automaton can start reading $b's$. From there, we recognize the $b's$ analogously.

Remark 1 There was a typo in the suggested problem. The handout said $L = \{a^{3n}b^{4n} : n, m \geq 0\}$ which does not make sense, since m is not used. If the language were $L = \{ a^{3n}b^{4n} \ : \ n \geq 0 \}$ then L would not be regular; it can be proven (using PL) by choosing the word $w = a^{3p}b^{4p}$ and later choosing $i = 0$. Complete the missing details as an exercise.

Problem 2: Consider $L = \{ a^nba^{4n} \ : \ n > 0 \}$. Is it regular or not? Prove it.

Solution: No, it is not. We'll prove it by using the Pumping Lemma.

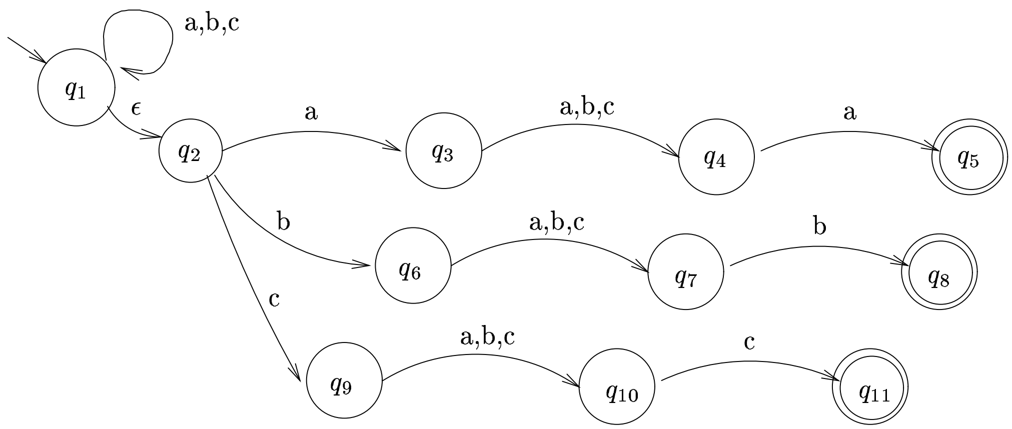
Assume by contradiction that L is regular. Then, by PL, we know that there exists a pumping length $p > 0$ such that any word $w \in L$, $|w| \geq p$, can be partitioned as $xyz = w$ (with $|xy| \leq p$ and $|y| > 0$) in such a way that, for any $i \geq 0$, the word xy^iz also belongs to L.

However, consider $w = a^pba^{4p}$. Clearly $|w| \geq p$. Moreover, any partition of w into xyz must be such that y comprises only $a's$ (since $|xy| \leq p$). Then $y = a^k$ for some $k > 0$ (since $|y| > 0$). Now, we consider the word $w' = xy^iz$ with $i = 0$, that is the word $w' = xy^0z = a^{p-k}ba^{4p}$. By PL, $w' \in L$. But $k > 0$ and therefore $4(p-k) \neq 4p$, which implies that $w' \notin L$. We obtain a contradiction.

Problem 3: Consider L="All strings in $\{ a, b, c \}^*$ that end with a palindrome of length 3". (*Palindromes* are the words that are read the same way from both ends, e.g. **atoyota**). Is it regular or not? Prove it.

Solution:

Yes, it is regular. The following is the NFA that recognizes it:



Note that the related language $L' = \{ w : w = w^R \}$ is not regular (you may check this on the book, see exercise 1.23(d)). The difference is that in order to check whether a word is palindrome or not, it is much easier to remember a word of fixed length (say 3) than an arbitrary long word. For language L, the automaton must remember words of length 3, whereas in L' words are of potentially unbounded length.

Problem 4: Consider $L = \{ a^n b^m : n \neq m \}$. Prove this language is not regular by using both

1. Closure properties of regular languages (eg. "if L were regular, then $L \cap R$ would be regular because we know R is" or "$\overline{L}$ would be also" or "L^R would be also" or "$L \cup R$ would be also, etc.
2. Pumping Lemma. This is tricky, but it definitely shows how the contradiction should come from any word partition. *Hint: notice that most choices of words do not work. Use the word $a^p b^{p+p!}$, where $p! = p \cdot (p-1) \cdots 2 \cdot 1$; use the fact that any number k, $1 \leq k \leq p$ divides $p!$ exactly.*

Solution:

Using closure properties of regular languages:

By contradiction, assume L is regular. Then $\overline{L}$ is also regular. Let's see how $\overline{L}$ looks like: $\overline{L}$ is the language of all the words that either (a) are of the form $a^n b^m$, where $n = m$, or (b) *contain* the substring ba (which is not allowed in L). Therefore, $\overline{L} = \{a^n b^n : n \geq 0\} \cup \{w : w = (a \cup b)^* ba (a \cup b)^*\}$. And thus,

$$\overline{L} \setminus \{ w \ : \ w = (a \cup b)^* ba (a \cup b)^* \} = \{ a^n b^n \ : \ n \geq 0 \} .$$

The left hand side of the last expression is regular because the *set-minus* operation is regular (recall that $A \setminus B = A \cap \overline{B}$). However, the right hand side is not. We've got a contradiction.

Using the pumping lemma:

Assume by contradiction that L is regular. Then, by PL, we know that there exists a pumping length $p > 0$ such that any word $w \in L$, $|w| \geq p$ can be partitioned as $xyz = w$ (with $|xy| \leq p$ and $|y| > 0$) in such a way that, for any $i \geq 0$, the word xy^iz also belongs to L.

However, we use the hint and consider $w = a^p b^{p+p!}$. Clearly $|w| \geq p$. Moreover, any partition of w into xyz must be such that y comprises only $a's$ (since $|xy| \leq p$). Then, it must be the case that $y = a^k$ for some $k > 0$ (since $|y| > 0$). Now, we consider the word $w' = xy^iz$, for some $i \geq 0$ which we leave unspecified for now. The word w' equals $xy^iz = a^{p+(i-1)k} b^{p+p!}$. We want to prove that *for any value of* k (that is, any possible y and thus, any possible partition) *there exists a value of* $i \geq 0$ *which causes* w' *to have the same number of* $a's$ *and* $b's$*:* $n = p + (i-1)k = p + p! = m$. This contradicts the condition $n \neq m$ for words in L.

Indeed, by solving $p + (i-1)k = p + p!$ we get $i = \frac{p!}{k} + 1$. So, for any value of k (recall that $0 \leq k \leq p$) $\frac{p!}{k} + 1$ will be a positive integer and thus, there exists a value $i \geq 0$ such that $w' \notin L$. Nevertheless, by PL, $w' \in L$. We've got a contradiction.