Discussion Section Notes – Solutions of Suggested Problems

Problem 1: Consider \( L = \{ a^{3n}b^{4m} : n, m \geq 0 \} \). Is it regular or not? Prove it.

Solution: Yes, it is regular. The following is the DFA that recognizes it:

![DFA Diagram](image)

Although the language \( L \) seems very similar to some nonregular languages like \( 0^n1^n \), there is one important difference: in this case there is no unbounded amount of information to remember. “Wait a second” – you may say – “Doesn’t the automaton have to remember the number of \( a \)'s in order to check if it is multiple of 3?” Not really. The automaton only needs to remember whether the number of initial \( a \)'s is a multiple of 3. This task can be done by creating one state for the case “number of \( a \)'s is exactly multiple of \( 3 \) (\( q_0 \)), another state for “number of \( a \)'s is one more than a multiple of \( 3 \) (\( q_1 \)), and another one for “number of \( a \)'s is two more than a multiple of \( 3 \) (\( q_2 \)). After that, it is easy. Only on state \( q_1 \) the automaton can start reading \( b \)'s. From there, we recognize the \( b \)'s analogously.

Remark 1 There was a typo in the suggested problem. The handout said \( L = \{ a^{3n}b^{4m} : n, m \geq 0 \} \) which does not make sense, since \( m \) is not used. If the language were \( L = \{ a^{3n}b^{4n} : n \geq 0 \} \) then \( L \) would not be regular; it can be proven (using PL) by choosing the word \( w = a^{3p}b^{4p} \) and later choosing \( i = 0 \). Complete the missing details as an exercise.

Problem 2: Consider \( L = \{ a^nba^{4n} : n > 0 \} \). Is it regular or not? Prove it.

Solution: No, it is not. We'll prove it by using the Pumping Lemma.

Assume by contradiction that \( L \) is regular. Then, by PL, we know that there exists a pumping length \( p > 0 \) such that any word \( w \in L \), \( |w| \geq p \), can be partitioned as \( xyz = w \) (with \( |xy| \leq p \) and \( |y| > 0 \)) in such a way that, for any \( i \geq 0 \), the word \( xyz^i \) also belongs to \( L \).

However, consider \( w = a^pbba^{4p} \). Clearly \( |w| \geq p \). Moreover, any partition of \( w \) into \( xyz \) must be such that \( y \) comprises only \( a \)'s (since \( |xy| \leq p \)). Then \( y = a^k \) for some \( k > 0 \) (since \( |y| > 0 \)). Now,
we consider the word \( w' = xy^i z \) with \( i = 0 \), that is the word \( w' = xy^0 z = a^{p-k}ba^p \). By PL, \( w' \in L \). But \( k > 0 \) and therefore \( 4(p - k) \neq 4p \), which implies that \( w' \not\in L \). We obtain a contradiction.

**Problem 3:** Consider \( L = \{ \text{All strings in \( \{a, b, c\}\}^* \) that end with a palindrome of length 3}\). (Palindromes are the words that are read the same way from both ends, e.g. atoyota). Is it regular or not? Prove it.

**Solution:**
Yes, it is regular. The following is the NFA that recognizes it:

![NFA Diagram]

Note that the related language \( L' = \{ w : w = w^R \} \) is not regular (you may check this on the book, see exercise 1.23(d)). The difference is that in order to check whether a word is palindrome or not, it is much easier to remember a word of fixed length (say 3) than an arbitrary long word. For language \( L \), the automaton must remember words of length 3, whereas in \( L' \) words are of potentially unbounded length.

**Problem 4:** Consider \( L = \{ a^n b^m : n \neq m \} \). Prove this language is not regular by using both

1. Closure properties of regular languages (eg. "if \( L \) were regular, then \( L \cap R \) would be regular because we know \( R \) is" or "\( L \) would be also" or "\( L^R \) would be also" or "\( L \cup R \) would be also", etc.

2. Pumping Lemma. This is tricky, but it definitely shows how the contradiction should come from any word partition. Hint: notice that most choices of words do not work. Use the word \( a^p b^{p+p!} \), where \( p! = p \cdot (p - 1) \cdots 2 \cdot 1 \); use the fact that any number \( k, 1 \leq k \leq p \) divides \( p! \) exactly.

**Solution:**
Using closure properties of regular languages:

By contradiction, assume \( L \) is regular. Then \( L \) is also regular. Let’s see how \( L \) looks like: \( L \) is the language of all the words that either (a) are of the form \( a^n b^m \), where \( n = m \), or (b) contain the
substring $ba$ (which is not allowed in $L$). Therefore, $\mathcal{T} = \{a^n b^n : n \geq 0 \} \cup \{ w : w = (a \cup b)^* ba (a \cup b)^* \}$. And thus,

$$\mathcal{T} \setminus \{ w : w = (a \cup b)^* ba (a \cup b)^* \} = \{ a^n b^n : n \geq 0 \}.$$ 

The left hand side of the last expression is regular because the set-minus operation is regular (recall that $A \setminus B = A \cap \overline{B}$). However, the right hand side is not. We’ve got a contradiction.

Using the pumping lemma:

Assume by contradiction that $L$ is regular. Then, by PL, we know that there exists a pumping length $p > 0$ such that any word $w \in L$, $|w| \geq p$ can be partitioned as $xyz = w$ (with $|xy| \leq p$ and $|y| > 0$) in such a way that, for any $i \geq 0$, the word $xy^iz$ also belongs to $L$.

However, we use the hint and consider $w = a^p b^{p+p!}$. Clearly $|w| \geq p$. Moreover, any partition of $w$ into $xyz$ must be such that $y$ comprises only $a$'s (since $|xy| \leq p$). Then, it must be the case that $y = a^k$ for some $k > 0$ (since $|y| > 0$). Now, we consider the word $w' = xy^iz$, for some $i \geq 0$ which we leave unspecified for now. The word $w'$ equals $xy^iz = a^{p+(i-1)k} b^{p+p!}$. We want to prove that for any value of $k$ (that is, any possible $y$ and thus, any possible partition) there exists a value of $i \geq 0$ which causes $w'$ to have the same number of $a$'s and $b$'s: $n = p + (i-1)k = p + p! = m$.

This contradicts the condition $n \neq m$ for words in $L$.

Indeed, by solving $p + (i - 1)k = p + p!$ we get $i = \frac{p!}{k} + 1$. So, for any value of $k$ (recall that $0 \leq k \leq p$) $\frac{p!}{k} + 1$ will be a positive integer and thus, there exists a value $i \geq 0$ such that $w' \not\in L$. Nevertheless, by PL, $w' \in L$. We’ve got a contradiction.