Problem 0 (1 point): PRINT your name _____________________________

Problem 1 (33 points): Let \( L_1 \) and \( L_2 \) be two languages over the alphabet \( \Sigma \). The shuffle of \( L_1 \) and \( L_2 \) is the set of all words that can be obtained shuffling a word \( w_1 \) from \( L_1 \) and a word \( w_2 \) from \( L_2 \), i.e., breaking each word into pieces \( w_1 = u_1 u_2 \ldots u_n \) and \( w_2 = v_1 v_2 \ldots v_n \) \((n \geq 1)\) and forming the word \( u_1 v_1 u_2 v_2 \ldots u_n v_n \). (Notice: the pieces \( u_i \) and \( v_i \) can be arbitrary words in \( \Sigma^* \), not just single characters. There is more than one way to break up a word into pieces. In the shuffle you have to consider all of them.)

For example if \( L_1 = \{aac\} \) and \( L_2 = \{dc,b\} \), then
\[
L_1 \| L_2 = \{aacde, aadce, aadec, adace, adaec, adace, daace, dace, aecac, aedac, acea, baac\}
\]

Prove that if \( L_1 \) and \( L_2 \) are regular languages, then also \( L_1 \| L_2 \) is regular.
[HINT: given DFAs \( M_1 = (Q_1, s_1, F_1, \Sigma, \delta_1) \) and \( M_2 = (Q_2, s_2, F_2, \Sigma, \delta_2) \) recognizing the languages \( L_1 \) and \( L_2 \), show how to build an NFA \( M = (Q, s, F, \Sigma, \delta) \) with states \( Q = Q_1 \times Q_2 \) that recognizes \( L_1 \| L_2 \)].

Problem 2 (33 points): A palindrome is a word that reads the same forward and backward. For example \( abacaba \) and \( ababbbaba \) are palindromes, but \( abab \) is not. Prove that the language \( L \) of all words over \( \Sigma = \{a, b\} \) that are not a palindrome is not regular.

Solution: Proof by contradiction. Assume \( L \) is regular. Then, by problem 4 in homework assignment 2, also the complement of \( L \) (\( \bar{L} \)) is regular. Notice that \( \bar{L} \) is the set of words that are palindromes. We now prove that \( \bar{L} \) is not regular using the pumpin lemma. Let \( n \) be the pumping length associated to language \( \bar{L} \) and consider the word \( w = a^n b a^n \). Notice that \( w \) is a palindrome, i.e., \( w \in \bar{L} \). Since \( w \) is longer than \( n \), we know from the pumping lemma that there exists strings \( x, y, z \) such that \( w = xyz \), \( xy < n \), \( y \neq \epsilon \) and \( \delta((q_1, q_2), a) \) for all \( a \neq \epsilon \), and \( \delta((q_1, q_2), \epsilon) = \emptyset \).
Problem 3 (33 points): In this problem you will prove that the language of all words that are not palindromes is context free. Let $P$ be the language of all palindromes over the alphabet $\Sigma = \{a,b\}$. Answer the following questions. (Part (b) and (c) are independent. You can solve them in any order.)

(a) Give a context free grammar that generates $P$.

Solution: The language $P$ is generated by the following grammar:

$$P \rightarrow \epsilon | aP | bP$$

(b) Convert the context free grammar from part (a) into an equivalent push down automaton.

Solution: The state diagram of the PDA corresponding to the grammar is

(c) Prove that the complement of $P$ (i.e., the set of all words that are not palindromes) is also context free by giving a CFG that generates $\Sigma^* - P$.

Solution: A possible solution is the following:

$$S \rightarrow aSa|bSb|aP|bPa$$

$$P \rightarrow \epsilon | aP | bP$$

where $P$ generates the set of palindromes as defined in part (a). An alternative solution is given by the grammar:

$$S \rightarrow aSa|bSb|aTa|bTa$$

$$T \rightarrow \epsilon | aT | bT$$

where $T$ generates the set of all strings.