Due: beginning of class on Fri., Mar 17, 2000

Problem 1
In class we proved that the language

\[ \text{HALT}_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w \} \]

is not decidable by reduction from the acceptance problem \( A_{TM} \).

In this problem you are asked to give a direct proof of the same result. Prove (using a diagonalization argument) that the language \( \text{HALT}_{TM} \) is undecidable.

Problem 2
In the second midterm you were asked to prove that the language

\[ L = \{ \langle G, E \rangle \mid G \text{ is a CFG and } E \text{ is a regular expression such that } \mathcal{L}(G) \subseteq \mathcal{L}(E) \} \]

In this problem you are asked to study what happens if you exchange \( E \) and \( G \) in the definition of \( L \). That is, consider the language

\[ L' = \{ \langle G, E \rangle \mid G \text{ is a CFG and } E \text{ is a regular expression such that } \mathcal{L}(E) \subseteq \mathcal{L}(G) \} \].

Is \( L' \) decidable? (Prove or disprove)

Problem 3
Do problem 5.21(b) from the book. I.e., show that the emptiness problem \( E_{2DFA} \) for two headed finite automata is undecidable. (See the textbook for precise definition of 2DFA and \( E_{2DFA} \).)

Also, answer the following questions: is \( E_{2DFA} \) enumerable? is \( E_{2DFA} \) co-enumerable? (Briefly justify your answer)
Problem 4

Let $L$ be a language.

(a) Show that if $L \leq_m \overline{L}$ (i.e., $L$ maps to the complement of $L$), then $L$ is either (1) decidable or (2) neither enumerable nor co-enumerable.

(b) The following statement is false:

‘If $L$ is decidable then $L \leq_m \overline{L}$’

Find a counterexample! i.e., find a decidable language $L_1$ such that $L_1$ does not map reduce to its complement. Now, find another counterexample! i.e. find a decidable language $L_2$ different from $L_1$ such that $L_2$ does not map reduce to $\overline{L_2}$. Finally, prove that $L_1$ and $L_2$ are the only two counterexamples! i.e., show that for any decidable language $L$ different from $L_1$ or $L_2$, it holds $L \leq_m \overline{L}$.

(c) Optional (This part won’t be graded): Can you find a non-decidable language such that $L \leq_m \overline{L}$? (Notice: from part (a) you know that $L$ is neither enumerable nor co-enumerable.)