INDUCTIVE STEP: Suppose that \( P(j) \) is true for all \( j \) with \( 1 \leq j \leq k \), that is, that the second player can always win whenever there are \( j \) matches where \( 1 \leq j \leq k \) in each of the two piles at the start of the game. Now suppose that there are \( k+1 \) matches in each of the two piles at the start of the game and suppose that the first player removes \( j \) matches \((1 \leq j \leq k)\) from one of the piles, leaving \( k+1-j \) matches in this pile. By removing the same number of matches from the other pile, player two creates the situation where there are two piles each with \( k+1-j \) matches. Because \( 1 \leq k+1-j \leq k \) the second player can always win by the induction hypothesis. We complete the proof by noting that if the first player removes all \( k+1 \) matches from one of the piles, the second player can win by removing all the remaining matches.

EXAMPLE 14

Show that if \( n \) is an integer greater than 1, then \( n \) can be written as the product of primes.

Solution: Let \( P(n) \) be the proposition that \( n \) can be written as the product of primes.

BASIS STEP: \( P(2) \) is true, since 2 can be written as the product of one prime, itself. [Note that \( P(2) \) is the first case we need to establish.]

INDUCTIVE STEP: Assume that \( P(j) \) is true for all positive integers \( j \) with \( j \leq k \). To complete the inductive step, it must be shown that \( P(k+1) \) is true under this assumption.

There are two cases to consider, namely, when \( k+1 \) is prime and when \( k+1 \) is composite. If \( k+1 \) is prime, we immediately see that \( P(k+1) \) is true. Otherwise, \( k+1 \) is composite and can be written as the product of two positive integers \( a \) and \( b \) with \( 2 \leq a \leq b \leq k+1 \). By the induction hypothesis, both \( a \) and \( b \) can be written as the product of primes. Thus, if \( k+1 \) is composite, it can be written as the product of primes, namely, those primes in the factorization of \( a \) and those in the factorization of \( b \).

Remark: Since 1 is a product of primes, namely, the empty product of no primes, we could have started the proof in Example 14 with \( P(1) \) as the basis step. We chose not to do this because many people find this confusing.

Note that Example 14 completes the proof of the Fundamental Theorem of Arithmetic, which asserts that every nonnegative integer can be written uniquely as the product of primes in nondecreasing order. We showed in Section 2.6 (see page 183) that an integer has at most one such factorization into primes. Example 14 shows there is at least one such factorization.

Using the principle of mathematical induction, instead of strong induction, to prove the result in Example 14 is difficult. However, as Example 15 shows, some results can be readily proved using either the principle of mathematical induction or strong induction.

EXAMPLE 15

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: We will prove this result using the principle of mathematical induction. Then we will present a proof using strong induction. Let \( P(n) \) be the statement that postage of \( n \) cents can be formed using 4-cent and 5-cent stamps.

We begin by using the principle of mathematical induction.

BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.

INDUCTIVE STEP: Assume that \( P(k) \) is true, so that postage of \( k \) cents can be formed using 4-cent and 5-cent stamps. If at least one 4-cent stamp was used, replace it with a 5-cent stamp to form postage of \( k+1 \) cents. If no 4-cent stamps were used, postage of \( k \) cents was formed using just 5-cent stamps. Since \( k \geq 12 \), at least three 5-cent stamps were used. So, replace three 5-cent stamps with four 4-cent stamps to form postage of \( k+1 \) cents. This completes the inductive step, as well as the proof by the principle of mathematical induction.

Next, we will use strong induction. We will show that postage of 12, 13, 14, and 15 cents can be formed and then show how to get postage of \( k+1 \) cents for \( k \geq 15 \) from postage of \( k \) cents.

BASIS STEP: We can form postage of 12, 13, 14, and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively.

INDUCTIVE STEP: Let \( k \geq 15 \). Assume that we can form postage of \( j \) cents, where \( 12 \leq j \leq k \). To form postage of \( k+1 \) cents, use the stamps for postage of \( k \) cents together with a 4-cent stamp. This completes the inductive step, as well as the proof by strong induction.

(There are other ways to approach this problem besides those described here. Can you find a solution that does not use mathematical induction?)

Remark: Example 15 shows how we can adapt strong induction to handle cases where the inductive step is valid only for sufficiently large values of \( k \). In particular, to prove that \( P(n) \) is true for \( n = j, j+1, j+2, \ldots \), where \( j \) is an integer, we first show that \( P(j), P(j+1), P(j+2), \ldots, P(k) \) are true (the basis step), and then we show that \( [P(j) \land P(j+1) \land P(j+2) \land \cdots \land P(k)] \rightarrow P(k+1) \) is true for every integer \( k \geq j \) (the inductive step). For example, the basis step of the second proof in the solution of Example 15 shows that \( P(12), P(13), P(14), P(15) \) are true. We need to prove these cases separately since the inductive step, which shows that \( [P(12) \land P(13) \land \cdots \land P(k)] \rightarrow P(k+1) \), holds only when \( k \geq 15 \).

THE WELL-ORDERING PROPERTY

The validity of mathematical induction follows from the following fundamental axiom about the set of integers.

THE WELL-ORDERING PROPERTY

Every nonempty set of nonnegative integers has a least element.

The well-ordering property can often be used directly in proofs.

EXAMPLE 16

Use the well-ordering property to prove the division algorithm. Recall that the division algorithm states that if \( a \) is an integer and \( d \) is a positive integer, then there are unique integers \( q \) and \( r \) with \( 0 \leq r < d \) and \( a = dq + r \).

Solution: Let \( S \) be the set of nonnegative integers of the form \( a - dq \) where \( q \) is an integer. This set is nonempty since \( -dq \) can be made as large as desired (taking \( q \) to be a negative integer with large absolute value). By the well-ordering property \( S \) has a least element \( r = a - dq \).

The integer \( r \) is nonnegative. It is also the case that \( r < d \). If it were not, then there would be a smaller nonnegative element in \( S \), namely, \( a - d(q_0 + 1) \). To see this, suppose that \( r \geq d \). Since \( a = dq_0 + r \), it follows that \( a - d(q_0 + 1) = (a - dq_0) - d = r - d \geq 0 \).