1. **Approximation algorithms** The following $n = 12$ jobs with given processing times have to be scheduled on $m = 3$ parallel and identical processors with the objective of minimizing the makespan. $C_j$ is the completion time of a job.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
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<td>9</td>
<td>12</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>10</td>
<td>8</td>
<td>11</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

(a) Draw the List-Scheduling-schedule as a Gantt-Chart. How much is $C_{max}$ and how much is $\sum C_j$?
(b) Draw the LPT-schedule as a Gantt-Chart. How much is $C_{max}$ and how much is $\sum C_j$?
(c) Draw the SPT-schedule as a Gantt-Chart. How much is $C_{max}$ and how much is $\sum C_j$?
(d) Find the optimal and the worst list concerning the objective of minimizing the makespan.
(e) Find the optimal and the worst list concerning the objective of minimizing the sum of the processing times.

2. **Approximation algorithms**

(a) Explain why $\sum_{j=1}^{n} p_j/m$ and $p_{max} = max\{p_j\}$ are *Lower Bounds* for the makespan of any algorithm solving $P||C_{max}$.
(b) Find *Upper Bounds* for the makespan of List Scheduling for $P||C_{max}$.

3. **Graphs and their representation** The first question refers to the following directed graph:

![Graph Diagram]

Answer the following questions:

(a) Give the degrees of the vertices $A$, $B$, and $E$.
(b) What are the direct neighbors of vertex $B$?
(c) Are there any loops in the graph?
(d) Are there any cycles in the graph? What about simple cycles?
(e) Find a trail from vertex $A$ to vertex $E$.
(f) Find a path from vertex $A$ to vertex $E$ that is no trail.
(g) Is this graph a simple directed graph? Explain why or why not.
(h) Does this graph have a source and/or a sink? Explain why or why not.
(i) Is this graph (strongly or weakly) connected or disconnected? Explain your answer.
(j) Give the adjacency matrix representation of the graph.
(k) Give the adjacency list representation of the graph.

4. **Graphs and their representation.** The second question refers to the following undirected graph:

![Graph](image)

Answer the following questions:

(a) Give the degrees of the vertices A, B, and E.
(b) What are the direct neighbors of vertex B?
(c) Are there any loops in the graph?
(d) Are there any cycles in the graph? What about simple cycles?
(e) Find a trail from vertex A to vertex E.
(f) Find a path from vertex A to vertex E that is no trail.
(g) Is this graph a simple undirected graph? Explain why or why not.
(h) Does this graph have a source and/or a sink? Explain why or why not.
(i) Is this graph (strongly or weakly) connected or disconnected? Explain your answer.
(j) Give the adjacency matrix representation of the graph.
(k) Give the adjacency list representation of the graph.

5. **Some graph properties:** Prove the following claims.

(a) Let \( G = (V, E) \) be an undirected graph with \(|E|\) edges. Then \( \sum_{v \in V} \deg(v) = 2|E| \).
(b) An undirected graph has an even number of vertices of odd degree.
(c) Let \( G = (V, E) \) be a directed graph with \(|E|\) edges. Then \( \sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E| \).

6. **Trees.** Draw a Binary Tree of height 3 with a maximum number of leaf nodes. Number the nodes alphabetically starting with A in the root and then level for level from the left to the right in increasing order.

(a) What is the parent of node L?
(b) What are the children of node C?
(c) Is the tree balanced? Explain your answer.
(d) Name the leaves of tree. How many leaves does this tree have? Can you generalize your answer?
(e) How many internal vertices does the tree have? Can you generalize your answer?

7. **Trees.** For each of the following statements, either prove it’s true or provide a counter example.

(a) Every tree with \( n \) nodes has exactly \( n - 1 \) edges.
(b) Every graph with exactly \( n - 1 \) edges is a tree.

8. Euler and Hamilton Tours.

(a) Describe briefly what the problem of the 7 Bridges of Königsberg is.

(b) What is an Euler tour? What is an Euler Cycle? What is a Hamilton Tour? What is a Hamilton Cycle? Is it (from a computationally point of view) harder to compute Hamilton Tours than to compute Euler Tours? Explain why or why not.

(c) Right or false? If a graph \( G \) has 3 or more even vertices \( \leftrightarrow \) \( G \) has no Euler tour.

9. Modelling problems with graphs: Hamiltonian/Euler. A ballet recital consists of some number of different acts, each containing several dancers. Suppose you are given a list of all the acts in the recital, with the names of the dancers that will appear in each act. Some dancers may be in more than one act, but each act requires a different costume. To allow the dancers time to change costumes, the recital should be set up so that no dancer is in two consecutive acts. Our goal is to find an ordering of the acts for the recital so that no dancer is in back-to-back acts.

(a) Describe how to model this situation using a graph. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

(b) Now that you’ve modeled the situation with a graph, which problem from graph theory are you trying to solve on this graph?

(c) Is it always possible to achieve the goal? Explain.

10. Modelling problems with graphs: Hamiltonian/Euler. We say a matrix has dimensions \( m \times n \) if it has \( m \) rows and \( n \) columns. If matrix A has dimensions \( x \times y \) and matrix B has dimensions \( z \times w \), then the product \( AB \) exists if and only if \( y = z \). In the case where the product exists, \( AB \) will have dimensions \( x \times w \). In this problem, we are given a list of matrices and their dimensions, and we want to determine if there is an order in which we can multiply all the matrices together, using each matrix exactly once. For example, here is a possible list of matrices and their dimensions:

\[
\begin{align*}
A & \text{ is } 3 \times 2 \\
B & \text{ is } 6 \times 3 \\
C & \text{ is } 2 \times 5 \\
D & \text{ is } 5 \times 3 \\
E & \text{ is } 3 \times 6 \\
F & \text{ is } 6 \times 2
\end{align*}
\]

(a) Given any list of matrices and dimensions, describe how to draw a graph so that each order in which we can multiply the matrices corresponds to a Hamiltonian tour of your graph. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

(b) Draw the graph described in part (a) for the example list of matrices given above.

(c) Given any list of matrices and dimensions, describe how to draw a graph so that each order in which we can multiply the matrices corresponds to an Eulerian tour of your graph. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

(d) Draw the graph described in part (c) for the example list of matrices given above.

(e) For the given example list of matrices, give one order in which we can multiply those matrices, or say that no such order exists.
11. **Modelling problems with graphs.** Say that two actors are co-stars if they have been in the same movie. Show that in any group of six actors, we can either find a group of three such that all pairs in the group are co-stars, or a group of three so that no two in the group are co-stars.