1 Scheduling

1. LPT Scheduling

   (a) Find the Upper Bound for makespan of LPT Scheduling for $P||C_{max}$.
   
   (b) Find a tight worst-case example for the makespan achieved by LPT Scheduling in comparison to the optimum makespan for $m = 5$ processors.
   
   (c) Prove that if optimal makespan of LPT schedule, $C_{max}^* < 3p_k$, then each processor has at most two jobs.
   
   (d) Prove that if each processor has at most two jobs, then LPT is always optimal.

2. Single Processor Scheduling

   (a) The following $n = 8$ jobs with given processing times have to be scheduled on a single processor with the objective of minimizing the maximum lateness $L_{max}$. $d_j$ is the maximum delay of each process.

   \[
   \begin{array}{cccccccc}
   p_j & A & B & C & D & E & F & G & H \\
   \hline
   p_j & 4 & 9 & 12 & 1 & 3 & 5 & 9 & 10 \\
   d_j & 6 & 13 & 8 & 1 & 4 & 3 & 7 & 10 \\
   \end{array}
   \]

   Calculate the values of $L_{max}$ and $\sum U_j$

   (i) If you schedule the jobs in the given order - A,B,...,H
   
   (ii) If you schedule the jobs using EDD algorithm
   
   (iii) If you schedule the jobs using the Moore’s algorithm
   
   (b) Prove that there is no better solution than the one obtained using Earliest Due Date (EDD) algorithm, to the problem of minimizing maximum lateness.
   
   (c) Prove that Moore’s algorithm provides the optimal solution for $1||\sum U_j$.

2 Graphs

3. You are given an undirected unweighted graph $G = (V, E)$, $|V| = n$, $|E| = m$. Determine, whether this graph contains a circuit or not.

   (a) Develop an algorithm that solves this problem.
   
   (b) Prove its correctness.
   
   (c) Find its time complexity.
4. There are $n$ cities on a plane with coordinates $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. All cities are distinct and located at different points on the plane. The city 1 has access to electricity. You need to connect cities with wires, such that all cities will have access to electricity. A city has access to electricity if it is city 1 or if it is connected with a wire to another city, that has access to electricity. The length of the wire that connects the two cities equals the euclidean distance between these cities. Your task is to connect the cities with wires, such that the total length of all wires is minimized.

(a) Develop an algorithm that solves this problem.
(b) Prove its correctness.
(c) Find its time complexity.

5. You have a chess board $n \times m$. There is a knight on cell $(x_s, y_s)$. The knight can make moves from cell $(x, y)$ to cells $(x + 1, y + 2), (x - 1, y + 2), (x + 1, y - 2), (x - 1, y - 2), (x + 2, y - 1), (x - 2, y - 1), (x + 2, y + 1)$ and $(x - 2, y + 1)$ (all cells must be on the board). Find the minimum number of steps the knight needs to make to get from cell $(x_s, y_s)$ to cell $(x_f, y_f)$.

(a) Develop an algorithm that solves this problem.
(b) Prove its correctness.
(c) Find its time complexity.

6. You are given an undirected connected weighted graph $G = (V, E)$, $|V| = n$, $|E| = m$. The weights of all edges are positive integer numbers. Your task is to delete edges in this graph so that the resulting graph will be connected and the product of all weights of all edges in this graph will be as small as possible.

(a) Develop an algorithm that solves this problem.
(b) Prove its correctness.
(c) Find its time complexity.

7. There is a set of $n$ courses. Each course has its prerequisites (from the same set). So course $i$ is assigned a set of courses that are required to be completed before course $i$ starts. You are given a set of $n$ courses, and each course is provided with a list of prerequisites. Find a schedule (a list) of courses, if such exists, such that all $n$ courses are in this list and they all can be completed. If there is no such a schedule, just output $-1$.

For example, let’s say we have 5 courses and a list $(A, D, C, B, E)$. We go from left to right and try to complete a current course. If all prerequisites for the current course are completed, then the current course can also be completed.

(a) Develop an algorithm that solves this problem.
(b) Prove its correctness.
(c) Find its time complexity.

8. You are given a rooted tree $G = (V, E)$, $|V| = n$, $|E| = m$. You have $q$ online queries $(u, v)$. For each query $(u, v)$ you need to answer whether $u$ is an ancestor of $v$ in the tree $G$.

(a) Develop an algorithm that solves this problem.
(b) Prove its correctness.
(c) Find its time complexity.

9. You are given a weighted directed graph $G = (V, E)$, $|V| = n$, $|E| = m$ with positive weights. You are also given a node $s$ in this graph. Find such node $v$, such that there is a path from $v$ to $s$ and that among all possible such nodes, the shortest path from $v$ to $s$ is as large as possible.

(a) Develop an algorithm that solves this problem.
(b) Prove its correctness.
(c) Find its time complexity.
3 Linear Programming

10. A company manufactures two products A and B. The company earns $3 for each piece of A and $2 for each piece of B. Raw materials V1 and V2 are necessary to build these products. But there are only 8 parts of V1 and 6 parts of V2 available. To manufacture single piece of product A, 2 parts of V1 and 1 part of V2 is required. To manufacture single piece of product B, 1 part of V1 and 1 part of V2 is required.

How many pieces of A and B does the company have to manufacture when the objective function is to maximize the earning?

(a) Formulate the mathematical program.
(b) Solve the problem graphically.
(c) Solve the problem with the Simplex algorithm.
(d) Give an interpretation of the dual values: If the company could have access to one more piece of V1, how would the production program and the earning change? If the company could have access to one more piece of V2, how would the production program and the earning change?