1. **Modelling problems with graphs: Hamiltonian/Euler.** A ballet recital consists of some number of different acts, each containing several dancers. Suppose you are given a list of all the acts in the recital, with the names of the dancers that will appear in each act. Some dancers may be in more than one act, but each act requires a different costume. To allow the dancers time to change costumes, the recital should be set up so that no dancer is in two consecutive acts. Our goal is to find an ordering of the acts for the recital so that no dancer is in back-to-back acts.

   (a) Describe how to model this situation using a graph. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

   (b) Now that you’ve modeled the situation with a graph, which problem from graph theory are you trying to solve on this graph?

   (c) Is it always possible to achieve the goal? Explain.

2. **Modelling problems with graphs: Hamiltonian/Euler.** We say a matrix has dimensions $m \times n$ if it has $m$ rows and $n$ columns. If matrix A has dimensions $x \times y$ and matrix B has dimensions $z \times w$, then the product $AB$ exists if and only if $y = z$. In the case where the product exists, $AB$ will have dimensions $x \times w$. In this problem, we are given a list of matrices and their dimensions, and we want to determine if there is an order in which we can multiply all the matrices together, using each matrix exactly once. For example, here is a possible list of matrices and their dimensions:

   \[
   \begin{align*}
   A & \text{ is } 3 \times 2 \\
   B & \text{ is } 6 \times 3 \\
   C & \text{ is } 2 \times 5 \\
   D & \text{ is } 5 \times 3 \\
   E & \text{ is } 3 \times 6 \\
   F & \text{ is } 6 \times 2
   \end{align*}
   \]

   (a) Given any list of matrices and dimensions, describe how to draw a graph so that each order in which we can multiply the matrices corresponds to a Hamiltonian tour of your graph. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

   (b) Draw the graph described in part (a) for the example list of matrices given above.

   (c) Given any list of matrices and dimensions, describe how to draw a graph so that each order in which we can multiply the matrices corresponds to an Eulerian tour of your graph. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

   (d) Draw the graph described in part (c) for the example list of matrices given above.

   (e) For the given example list of matrices, give one order in which we can multiply those matrices, or say that no such order exists.
3. **Modelling problems with graphs.** Say that two actors are co-stars if they have been in the same movie. Show that in any group of six actors, we can either find a group of three such that all pairs in the group are co-stars, or a group of three so that no two in the group are co-stars.

4. **Modelling problems with graphs: Topological Sorting.** Imagine you would have to perform in a project jobs $A, B, C, D, E, F$ with the following time dependencies:
   - $B$ must be finished before you can start with $D, E, F$.
   - $C$ must be finished before you can start with $B, E$.
   - $D$ must be finished before you can start with $A, E$.
   - $E$ must be finished before you can start with $A$.
   - $F$ must be finished before you can start with $D$.

   (a) Draw the corresponding graph to that problem (Why do you have to draw a directed graph?).
   (b) When you want to answer if all of the time dependencies can be fulfilled: what graph theoretic problem do you have to solve? Apply the corresponding algorithm to that problem. In what order will you have to perform the jobs?

5. **Single source shortest path.** After watching the Olympics live in Rio de Janeiro, Carmen Sandiego has suddenly decided to steal a few of the last 8 Olympic venues. The venues on her hit list are as follow: Rio de Janeiro (2016), London (2012), Beijing (2008), Athens (2004), Sydney (2000), Atlanta (1996), Barcelona (1992), and Seoul (1988). Because she’s not only a miracle thief, a student of computer science, she knows better than to try to find the optimal way in which to steal all 8 venues doing the least travelling. After all, thieves seldom resort to brute force. Instead, to make a game of it, she made some restrictions, somewhat based on location, for the order of the cities she’d visit (starting from Rio):
   - Sydney before Seoul, Beijing, and Athens
   - London, Seoul, and Barcelona before Atlanta
   - Beijing and Athens before London and Barcelona
   - Seoul before Athens

   (a) Draw a graph that corresponds to the problem.
   (b) Provide a topological sorting of the problem.
   (c) Given the following distances, use Bellman’s algorithm to find the shortest paths to all the other seven cities from Rio.

<table>
<thead>
<tr>
<th></th>
<th>Rio (2016)</th>
<th>London</th>
<th>Beijing</th>
<th>Athens</th>
<th>Sydney</th>
<th>Atlanta</th>
<th>Barcelona</th>
</tr>
</thead>
<tbody>
<tr>
<td>London (2012)</td>
<td>5771</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beijing (2008)</td>
<td>10775</td>
<td>5064</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Athens (2004)</td>
<td>6046</td>
<td>1488</td>
<td>4738</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sydney (2000)</td>
<td>8411</td>
<td>10571</td>
<td>5588</td>
<td>9534</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Atlanta (1996)</td>
<td>4776</td>
<td>4211</td>
<td>7185</td>
<td>5676</td>
<td>9297</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Barcelona (1992)</td>
<td>5312</td>
<td>709</td>
<td>5478</td>
<td>1188</td>
<td>10887</td>
<td>4574</td>
<td></td>
</tr>
<tr>
<td>Seoul (1988)</td>
<td>11280</td>
<td>5510</td>
<td>592</td>
<td>5297</td>
<td>5181</td>
<td>7121</td>
<td>5973</td>
</tr>
</tbody>
</table>
(d) Instead of her previous rules, Carmen Sandiego decided upon a new requirement: she wouldn't steal any venues from one decade ('00 - '09) at least one from the previous decade was already in her possession (for example, she must have one of either Rio or London before any of the others). Draw a graph that corresponds to this. (Hint: There should be cycles, starting point is still Rio).

(e) Use Dijkstra's algorithm to compute the shortest path from Athens to Seoul (using the graph from part d). Can Dijkstra's algorithm be applied to any network? Why or why not?

(f) Use Bellman-Ford's algorithm to compute the shortest path from Rio to all the other cities (using the graph from part d).

(g) What is the worst-case time complexity of the Bellman-Ford algorithm? Explain your answer.

6. **Knapsack problem.** The Pink Panther’s broken into a safe, but unfortunately he packed the wrong knapsack. His particular magical knapsack can only carry 4 kilograms. Before him are 1 gold bar, 2 sapphire crystals, and one priceless laminated cheat sheet. The gold bar weighs 2 kg, the sapphire crystals are 1 kg each, and the cheat sheet is 3 kg. Each gold bar is worth $1.8 million, the sapphire crystals are worth $1.1 million, and the priceless cheat sheet is worth a pricey $3 million. What should our feline thief steal to maximize his profit?

(a) Solve this problem with Complete Enumeration. How many vertices does the solution tree have?

(b) Give the smallest upper bound you can reasonably find before you start to decide if an object should or shouldn't be packed.

(c) What is the worst-case time complexity of Knapsack problems and why?

7. **Assignment problem.** Linus, Lucy, Rerun, and Sally are tasked with preparing for the annual talent show. They must (A) Clean the stage, (B) Decorate the gym, (C) Sell at least 30 tickets, and (D) Put on the first act. Of course, each of them have their own talents and are thus are able to perform the tasks at different speeds. How should they distribute the tasks such that they can finish as soon as possible given each person must perform exactly one task and all the tasks need to be finished? Below is the table that describes how fast they each can perform each task (days). The jobs cannot be done in parallel.

<table>
<thead>
<tr>
<th></th>
<th>Cleaning</th>
<th>Decorating</th>
<th>Coercing</th>
<th>Performing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Rerun</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>Linus</td>
<td>7</td>
<td>8</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>Lucy</td>
<td>11</td>
<td>13</td>
<td>1</td>
<td>15</td>
</tr>
</tbody>
</table>

(a) Solve this problem with Complete Enumeration. How many vertices does the solution tree have?

(b) Solve this problem with Branch-and-Bound. How many vertices does the solution tree have?

(c) What is the worst-case time complexity of Assignment problems and why?

8. **Counting.**

(a) How many different initials can a person have if they have a first name, a middle name, and a last name? (Hint: objects are A, B, C, ..., Z).

(b) A book publisher has 200 copies of Rosen. How many ways are there to store these books in their three warehouses? (Hint: objects are the 200 books).

(c) How many different words can be made by rearranging the letters in the word RUMPELSTILTSKIN (Hint: objects are the letters)?
(d) In a twist of fate, the final becomes 50 True/False problems, 15 of which are True. How many different answer keys can there be? (Hint: objects are the location of either the Trues or the Falses).

(e) Fourteen people show up to play for a community baseball game. How many different combinations of teams can be made? Baseball is played with 9 people. (Hint: the object (players) are numbered 1,2,...9).

(f) How many bits does it take to store someone’s initials in a database, assuming that each person has a first name, a middle name, and a last name? Simplify your answer.

(g) How many 8-bit binary strings are there that have at least 6 consecutive 1s (Hint: objects are 0, 1)?

9. Counting. For this problem, let $G_{m,n}$ represent the graph that looks like an $m \times n$ grid. For example, this is the graph $G_{3,6}$ which looks like a grid with 3 rows and 6 columns. The questions that follow are about $G_{m,n}$ for general $m$ and $n$, and all answers should be justified.

(a) What is the length of the shortest path from the bottom left corner to the top right corner of $G_{m,n}$?

(b) How many paths of this length are there?

(c) How many of these paths start with a north step?

(d) How many start with an east step?

(e) The total number of paths $p$ (part b) is clearly equal to the number of paths that start with a north step $n$ (part c) and the number of paths that start with an east step $e$ (part d). Give an algebraic proof that $p = n + e$. 

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