CSE 21
Math for Algorithms and Systems Analysis
Lecture 17
Constructing Recurrence Relations,
Solving 2nd order recurrences,
Recursion and Probability
Outline

• Determining recurrence relations
• Review of solving first-order linear recurrences
• Solving second-order linear recurrences
• Recursion and probability
Constructing Recursion Relations

• Given a problem use divide and conquer (break the problem down into smaller cases)
• If one of the subcases does not match the original recurrence, define a new recurrence
Review: Arrangements of Blue and Red Tiles

• Suppose we want to create a sequence of blue and red tiles such that no blue tile is in between two red tiles

• How many sequences of length $n$ are there that obey this constraint?
Define The Solution Recursively

- \( f(n) \): number of length \( n \) sequences of R and B that do not contain RBR
- \( g(n) \): number of length \( n \) sequences of R and B that do not contain RBR or start with BR

\[
\begin{align*}
  f(n) &= f(n-1) + g(n-1), \quad f(1) = 2, \quad f(2) = 4, \quad f(3) = 7 \\
  g(n) &= g(n-1) + f(n-2), \quad g(1) = 2, \quad g(2) = 3
\end{align*}
\]
More Complicated Towers of Hanoi

- Additional Movement Constraints (only moves between A and B, B and C, and C and A are allowed)
Recurrence 1

Necessary Configuration

A

Goal Peg

B

C
Workspace – Recurrence 2

Necessary Configuration 1

Necessary Configuration 2

Goal Peg

A
B
C

A
B
C
Workspace

Necessary Configuration 3

Goal Peg

A

B

C
Finding Closed form Expressions for Recursive Formulas

• Suppose we have the following recursive formula

\[ a(n) = 3a(n - 1) + 2, \quad a(0) = 1 \]

• Let’s compute some values

<table>
<thead>
<tr>
<th>n</th>
<th>a(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>53</td>
</tr>
<tr>
<td>4</td>
<td>161</td>
</tr>
</tbody>
</table>
Solving First-Order Linear Recurrences

• Given a recursive formula of the form

\[ a(n) = ba(n - 1) + c \]

• A non-recursive formula is given as

\[ a(n) = \left( a(0) - \frac{c}{1 - b} \right) b^n + \frac{c}{1 - b} \]
Second-Order Linear Recurrences

• Example:

\[ F(n) = F(n - 1) + F(n - 2), \quad F(0) = 0, \quad F(1) = 1 \]

• How can we determine a closed form formula for the nth Fibonacci number?
Second-Order Linear Recurrences

Second Case

\[ f(n) = af(n - 1) + bf(n - 2), \quad f(0) = f_0, \quad f(1) = f_1 \]

- In order to find a solution we have to first construct the characteristic polynomial

\[ x^2 - ax - b \]

- Let \( r_1 \) and \( r_2 \) be roots of the characteristic polynomial

\[ r_1^2 - r_1 x - b = 0 \quad r_2^2 - r_2 x - b = 0 \]

- Claim if \( r_1 = r_2 \)

\[ f(n) = K_1 r_1^n + K_2 n r_1^n \]
Inductive Proof (Base Cases Will be Solved Later in the Lecture)

• Inductive Step

Assume: \( f(n - 1) = K_1 r_1^{n-1} + K_2 (n - 1) r_1^{n-1} \)
\( f(n - 2) = K_1 r_1^{n-2} + K_2 (n - 2) r_1^{n-2} \)

Goal: \( f(n) = K_1 r_1^n + K_2 n r_1^n \)

\[
0 = f(n) - af(n - 1) - bf(n - 2)
\]
\[
= K_1 r_1^n + K_2 n r_1^n - a (K_1 r_1^{n-1} + K_2 (n - 1) r_1^{n-1})
- b (K_1 r_1^{n-2} + K_2 (n - 2) r_1^{n-2})
\]
\[
= K_1 r_1^{n-2} (r_1^2 - ar_1 - b) + K_2 n r_1^{n-2} (r_1^2 - ar_1 - b)
+ r_1^{n-2} K_2 (ar_1 + 2b)
\]
Inductive Proof Continued

\[ = r_{1}^{n-2}K_{2} (ar_{1} + 2b) \]

Our proof appears to have failed!

\[ x^2 - ax - b = (x - r_{1})^2 \]
\[ = x^2 - 2r_{1}x + r_{1}^2 \]
\[ a = 2r_{1} \]
\[ b = -r_{1}^2 \]

Substituting back in

\[ = r_{1}^{n-2}K_{2} ((2r_{1})r_{1} + 2(-r_{1}^2)) \]
\[ = 0 \]
Second-Order Linear Recurrences

\[ f(n) = af(n - 1) + bf(n - 2), \quad f(0) = f_0, f(1) = f_1 \]

- In order to find a solution we have to first construct the characteristic polynomial
  \[ x^2 - ax - b \]
- Let \( r_1 \) and \( r_2 \) be roots of the characteristic polynomial
  \[ r_1^2 - r_1 x - b = 0 \quad r_2^2 - r_2 x - b = 0 \]
- Claim if \( r_1 \neq r_2 \)
  \[ f(n) = K_1 r_1^n + K_2 r_2^n \]
Inductive Proof (Base Cases Will be Solved Later in the Lecture)

• Inductive Step:

Assume:

Goal:
Workspace
How do we find the values for the constants $K_1$ and $K_2$?

Example: $a(n) = 4a(n-1) - 4a(n-2)$, $a(0) = 3$, $a(1) = 8$

$$x^2 - 4x + 4 = (x - 2)^2$$

$$a(n) = K_12^n + K_2n2^n$$

$$a(0) = K_12^0 + K_2(0)2^0 = K_1 = 3$$

$$a(1) = (3)2^1 + K_2(1)2^1 = 6 + 2K_2 = 8$$

$$K_2 = 1$$

$$a(n) = (3)2^n + n2^n$$

Remember to check your answer!
Example Non-Repeated Root

- Example: \( a(n) = 6a(n-1) - 5a(n-2) \), \( a(0) = 3 \), \( a(1) = 8 \)

\[
x^2 - 6x + 5 = (x - 3)(x - 2)
\]

\[
a(n) = K_13^n + K_22^n
\]

\[
a(0) = K_1 + K_2 = 3
\]

\[
a(1) = 3K_1 + 2K_2 = 8
\]

\[
3(K_1 + K_2) - 3K_1 + 2K_2 = 3(3) - 8
\]

\[
K_2 = 1 \quad 1 + K_2 = 3, \quad K_2 = 2
\]

\[
a(n) = 3^n + 2(2)^n
\]

Remember to check your answer!
Exercise

• Find a closed form solution for the following second-order linear recurrence

\[ f(n) = f(n - 1) + f(n - 2), \quad f(0) = 0, \quad f(1) = 1 \]
Workspace
Example Tiling Problem

• How many ways can a strip of 1 by n squares be tiled using tile of size 1 by 1 and 1 by n

Two possible tilings for n = 6
Let’s develop a Recurrence for this Problem
Example of Recursion and Decision Trees

• A fair coin is flipped repeatedly until 2 consecutive head occur
• What is the expected number of total flips?
• Random Variable $X$ represents the number of flips before 2 consecutive heads occur
• We will also define two new random variables
Recursion and Decision Trees

Expected Future Flips $E[X]$

H

Total Flips = 2

T

Total Flips so Far 2, Expected Future Flips $E[X]$

T

Total Flips so Far 1, Expected Future Flips $E[X]$

H
What is $E[X]$?

\[
E[X] = \frac{1}{2} (1 + E[X]) + \frac{1}{4} (2 + E[X]) + \frac{1}{4} (2)
\]

\[
= \frac{1}{2} + \frac{1}{2}E[X] + \frac{1}{2} + \frac{1}{4}E[X] + \frac{1}{2}
\]

\[
= \frac{3}{2} + \frac{3}{4}E[X]
\]

\[
E[X] = 6
\]
Recursion and Probability

- Rock Paper Scissors (outcome matrix)

<table>
<thead>
<tr>
<th>Player 1 Selects</th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>Tie (redo)</td>
<td>P2 Wins</td>
<td>P1 Wins</td>
</tr>
<tr>
<td>Paper</td>
<td>P1 Wins</td>
<td>Tie (redo)</td>
<td>P2 Wins</td>
</tr>
<tr>
<td>Scissors</td>
<td>P2 Wins</td>
<td>P1 wins</td>
<td>Tie (redo)</td>
</tr>
</tbody>
</table>
Player’s Strategy

Player 1 Strategy:
P(rock) = \frac{1}{2}
P(scissors) = \frac{1}{4}
P(paper) = \frac{1}{4}

Player 2 Strategy:
P(rock) = \frac{1}{3}
P(scissors) = \frac{1}{3}
P(paper) = \frac{1}{3}
What is the Probability that Player 1 Wins?