CSE 21
Math for Algorithms and Systems Analysis
Lecture 14
Review of Random Variables
Outline

- Probability spaces
- Definition of a random variable
- Distribution function of a random variable
- Expected Value of a Random Variable
- Linearity of Expectations
- Independence of two random variables
- Variance
Probability Space

• A fair coin is flipped three times and the results of the flips are written as a list of length 3 consisting of Heads and Tails.

• Below is the set of all lists that could occur:

  (Tails, Tails, Tails) \hspace{1cm} (Tails, Tails, Heads)
  (Tails, Heads, Tails) \hspace{1cm} (Tails, Heads, Heads)
  (Heads, Tails, Tails) \hspace{1cm} (Heads, Tails, Heads)
  (Heads, Heads, Tails) \hspace{1cm} (Heads, Heads, Heads)

• We call the set of these lists the sample space. We refer to the sample space using the letter \( U \).
Probability Space Continued

• Given the sample space from the previous slide we may wish to answer the following questions
  – What is the probability that three heads occur?
  – What is the expected number of heads?
  – What is the probability that more heads occur than tails?
• Each of these questions requires that we specify the probability that each of the elements in the sample space occurs. We do this by defining a probability function $f$ (the domain is $U$ and the range is the real numbers).
Sample Probability Function

(Tails, Tails, Tails)
(Tails, Tails, Heads)
(Tails, Heads, Tails)
(Tails, Heads, Heads)
(Heads, Tails, Tails)
(Heads, Tails, Heads)
(Heads, Heads, Tails)
(Heads, Heads, Heads)

1/8
Example 2

• A biased coin (probability of heads = 2/3) is flipped 3 times.

• What is our sample space?
  – It is the same as before. All possible lists of length 3 consisting of Heads and Tails

(Tails, Tails, Tails) (Tails, Tails, Heads)
(Tails, Heads, Tails) (Tails, Heads, Heads)
(Heads, Tails, Tails) (Heads, Tails, Heads)
(Heads, Heads, Tails) (Heads, Heads, Heads)
What is the Probability Function?

- $(Tails, Tails, Tails)$: $(1/3)^3$
- $(Tails, Tails, Heads)$: $(1/3)^2(2/3)^1$
- $(Tails, Heads, Tails)$: $(1/3)^1(2/3)^2$
- $(Tails, Heads, Heads)$: $(2/3)^3$
- $(Heads, Tails, Tails)$
- $(Heads, Tails, Heads)$
- $(Heads, Heads, Tails)$
- $(Heads, Heads, Heads)$
Probability of an Event

• An event is defined as a subset of the sample space (e.g. the event that more heads come up than tails)

• The probability of an event is given by

\[ P(E) = \sum_{x \in E} f(x) \]
Example of Computing the Probability of an Event

• Let \( E \) be the event that more heads come up than tails

\[ E = \{(\text{Heads, Heads, Heads}), \\
(\text{Heads, Heads, Tails}), \\
(\text{Heads, Tails, Heads}), \\
(\text{Tails, Heads, Heads})\} \]

\[
P(E) = f((\text{Heads, Heads, Heads})) + f((\text{Heads, Heads, Tails})) \\
+ f((\text{Heads, Tails, Heads})) + f((\text{Tails, Heads, Heads}))
\]

\[
= \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)
\]
Problem

• Let $E$ represent the event that at least two consecutive heads or two consecutive tails occur in the 3 flips. What is $P(E)$?
Random Variables

• Events allow us to express binary properties of random processes
• Random variables will allow us to express continuous real-valued properties of a random processes
• Given a probability space \((U, f)\) a random variable is a function from \(U\) to the real numbers
X is a random variable specifying how many heads occurred in a sequence of 3 flips of our biased coin.
Distribution Function of a Random Variable

• We use the notation $f_X$ to represent the distribution function of the random variable $x$

• The domain of the distribution function is the real numbers. The range is the real numbers between 0 and 1 inclusive

• E.g. $f_X(2) = f((\text{Heads, Heads, Tails})) + f((\text{Heads, Tails, Heads})) + f((\text{Tails, Heads, Heads}))$
  
  $= (2/3)^2(1/3) + (2/3)^2(1/3) + (2/3)^2(1/3) = 3(2/3)^2(1/3)$
The Distribution Function of the Random Variable X

0

(1/3)^3

1

(2/3)(1/3)^2

2

(2/3)^2(1/3)^1

3

(2/3)^3

All other real numbers

0
Example 2 For Random Variables

• A number is selected uniformly at random from the set \( S = \{1,2,3,4,5,6,7\} \)
• Let \( X \) represent the outcome of a bet. If the number is odd the gambler wins $2 if it is not odd the gambler loses $1
• Problems:
  – Define the sample space, probability function, random variable \( X \), and distribution function of the random variable \( X \)
Workspace
Workspace
Expected Value of a Random Variable

• The expected value of a random variable is also often called the mean of a random variable
• We use the notation $E[X]$ to denote the expected value of the random variable $X$
• There are two formulas for the expected value (we will almost always use the second one)

\[
E[X] = \sum_{i \in U} X(i) f(i) \quad E[X] = \sum_{r \in \mathbb{R}} r f_X(r)
\]

• Question for the class: how do we arrive at the second formula from the first?
Defining a Random Variable in Terms of other Random Variables

• What does it mean to square a random variable?
• What does it mean to add two random variables together?
• Remember random variables are just functions
• Just as I can define one function in terms of another I can define one random variable in terms of another
Familiar Example

• Let $f(x) = \sin(x)$
• Let $g(x) = f(x)^2$
• What is $g(x)$?

• In order to define a random variable all we need to do is define a function from $U$ to the real numbers
• Let $X$ be a random variable
• Let $Z$ be a random variable that is the square of $X$
• For all elements $y \in U$, $Z(y) = X(y)^2$
Writing a Random Variable as the Sum of other Random Variables

• I flip a biased coin 3 times (probability of Heads is 2/3)
• Let X be a random variable that specifies how many heads occurred in the 3 flips
• $X_1$ is a random variable that takes on value 1 if and only if the $1^{st}$ flip was a heads
• $X_2$ is a random variable that takes on value 1 if and only if the $2^{nd}$ flip was heads
• $X_3$ is a random variable that takes on value 1 if and only if the $3^{rd}$ flip was heads
Writing a Random Variable as the Sum of other Random Variables

• What does it mean when I write?

\[ X = X_1 + X_2 + X_3 \]

• All it means is that the function \( X \) is equal to the sum of the functions \( X_1, X_2, X_3 \)

• Exercise: show that this equality is true by demonstrating that the function on the left side is equal to the function on the right side (what does it mean for two functions to be equal?)
Workspace
Linearity of Expectations

• Given a collection of random variables $X_1 \ldots X_n$
• $E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$
• This holds for any finite size collection of random variables
Linearity of Expectations Example

- We have just shown that for a sequence of 3 coin flips of a biased coin (P(Heads) = 2/3) that
  \[ X = X_1 + X_2 + X_3 \]
- By linearity of Expectations
  \[ E[X] = E[X_1] + E[X_2] + E[X_3] \]
- So if we are interested in computing E[X] we can just compute E[X_1], E[X_2], and E[X_3] and then add them up
Exercise Compute $E[X_1]$, $E[X_2]$, and $E[X_3]$

• First define the distribution function of $X_1$, $X_2$, and $X_3$

• Next apply the formula for expected value
Workspace
Linearity of Expectations More Complicated

• In a room of 10 sales people and 5 engineers each person selects uniformly at random another person to give his or her business card to
• Let $X$ be the number of engineers that gave their business cards to a salesperson
• What is $E[X]$?
• Let $Y$ be the number of salespeople that gave their business cards to an engineer
• What is $E[Y]$?
• What is $E[X + Y]$?
Workspace
Workspace
Independence of two Random Variables

- Given two random variables $X$ and $Y$ we define the joint distribution function $f_{X,Y}(u,v)$ which specifies the probability that random variable $X$ takes on value $u$ and random variable $Y$ takes on value $v$
- We flip a biased coin 3 times ($P(\text{Heads}) = 2/3$)
- $X = \# \text{ of Heads}$
- $Y = \# \text{ of Tails}$
- Define the joint distribution function $f_{X,Y}$
Workspace

\[ f_{X,Y}(0,0) = \]
\[ f_{X,Y}(0,1) = \]
\[ f_{X,Y}(0,2) = \]
\[ f_{X,Y}(0,3) = \]
\[ f_{X,Y}(1,0) = \]
\[ f_{X,Y}(1,1) = \]
\[ f_{X,Y}(1,2) = \]
\[ f_{X,Y}(1,3) = \]
\[ f_{X,Y}(2,0) = \]
\[ f_{X,Y}(2,1) = \]
\[ f_{X,Y}(2,2) = \]
\[ f_{X,Y}(2,3) = \]
\[ f_{X,Y}(3,0) = \]
\[ f_{X,Y}(3,1) = \]
\[ f_{X,Y}(3,2) = \]
\[ f_{X,Y}(3,3) = \]
Independence of two Random Variables

• Two random variables are independent iff

\[ f_{X,Y}(i, j) = f_X(i) f_Y(j) \quad \forall \, i, j \in \mathbb{R} \]

• Are X and Y independent random variables?
Variance

• There are two formulas for the variance of a random variable $X$

\[ \text{Var}[X] = E[(X - E[X])^2] \]

\[ \text{Var}[X] = E[X^2] - E[X]^2 \]
Variance Example

• Three cards are selected at random from a deck the standard deck
• Let X be the number of clubs selected
• What is Var[X]?
Workspace (hint define the distribution function of X)
Workspace (Hint: compute $E[X^2]$ and $E[X]^2$)
Variance of the sum of independent random Variables

• Given $X_1, \ldots, X_n$ with all pairs of random variables independent and $X = X_1 + \ldots + X_n$
• $\text{Var}[X] = \text{Var}[X_1] + \ldots + \text{Var}[X_n]$
• Exercise:
  – A 6-sided die is rolled 10 times
  – Let $X$ be the number of times an odd number comes up
  – What is $\text{Var}[X]$?
Workspace (write $X$ in terms of indicator r.v.s and show pair is independent)
Workspace (Use linearity of variances)
Additional Problems (not covered in lecture)

• A random number is selected uniformly at random from 0 to 99. Let X be the number of times the digit 9 appears in the number. What is E[X]? What is Var[X]?

• Three 6-sided dice are rolled. Let X be the number of unique numbers that appear on the dice. What is E[X] and Var[X]?

• In a class of n people let X be the number of subsets of two people that share the same birthday. What is Var[X]? (hard)
Additional Problems (not covered in lecture)

• 3 balls are selected at random from an urn containing 5 red balls and 3 green balls. Let $X$ be the number of red balls selected. What is $E[X]$ and $\text{Var}[X]$?

• A random permutation is selected on the set $\{1,2,3\}$. Let $X$ be the order of the permutation. What is $E[X]$ and $\text{Var}[X]$? (hard)