CSE 21
Math for Algorithms and Systems Analysis
Lecture 12
Expected Value of a Random Variable
Outline

• Review of Definition of Random Variables
• Expectation and Linearity of Expectations
Random Variables

• Given a probability space \((U, f)\)
  – Recall: \(U\) is the sample space (all elementary events that can occur), \(f\) is the probability function that specifies how probable each elementary event is

• A random variable is a function from \(U\) to the real numbers
Examples of Random Variables

• Sample space is the set of all 5 card hands from the standard deck of 52 cards
• The probability function is uniform over all hands (i.e. $f(y) = \frac{1}{\binom{52}{5}}$ for all hands $y$)
• We define the random variable $X$ to be the number of Queens in the 5-card hand
What does this function look like?

U (space of all 5-card hands)
Distribution of a Random Variable

• Given a sample space $U$, a probability function $f$, and a random variable $X$ we define a new function $f_X : \mathbb{R} \rightarrow \mathbb{R}$ that defines how probable it is for the random variable to take on any particular value.

$$f_X(y) = \sum_{t \in U \text{ and } X(t)=y} f(t)$$
Example of a Distribution of a Random Variable

• Sample space \( \{H, T\}^3 \) corresponding to flipping a fair coin 3 times (f(y) = 1/8 (i.e. all sequences of H and T are equally likely))

• Random variable X is the number of H’s in an elementary event (sequence of 3 flips)

\[
f_X(y) = \begin{cases} 
(\frac{1}{2})^3 \binom{3}{0}, & y = 0 \\
(\frac{1}{2})^3 \binom{3}{1}, & y = 1 \\
(\frac{1}{2})^3 \binom{3}{2}, & y = 2 \\
(\frac{1}{2})^3 \binom{3}{3}, & y = 3 \\
0, & \text{else}
\end{cases}
\]
Random Variable with Binomial Probability Distribution

• Given a sequence of n trials each with success probability q. We have seen before that

\[
P(\text{exactly } k \text{ successes}) = \binom{n}{k} q^k (1 - q)^{n-k}
\]

Define a random variable over the sample space of all sequences. The output of the random variable on a particular element of the sample space is the number of successes for that sequence.

• This random variable has what is known as a binomial probability distribution. That is, for any non-negative integer k we have:

\[
f_X(k) = \binom{n}{k} q^k (1 - q)^{n-k}
\]
Expected Value

• Sometimes it is useful to compute the expected value (or mean) of a random variable.

• Examples
  – Let $X$ be a random variable that represents the amount of profit we make on an investment. We may want to know our expected profit (or loss).
  – Let $X$ be a random variable representing the number of people that will show up at party you are throwing. You may want to know the expected number of party guests.
The expectation operator $E$ applied to a random variable $X$ is:

$$E[X] = \sum_{i \in U} X(i) f(i)$$

- Sum over all elements of the sample space
- Value of the random variable at that elementary event
- Probability of that elementary event
Alternative (and often more useful) formula for the Expectation of a Random Variable

By collecting elements of the sample space where the random variable has the same output, we can compute the expectation by summing over all values the random variable can take on and multiply by the distribution function of the random variable

\[ E[X] = \sum_{r \in \mathbb{R}} r f_X(r) \]

- Loop over all real numbers
- Random variable takes on value r
- Probability of random variable X taking on this value
Question

• We toss a biased coin 5 times (probability of heads is .8)
• What is the expected number of heads in the sequence of flips?
• Hints:
  – Define a random variable $X$ that indicates the number of heads in the sequence
  – Observe that this random variable has a binomial distribution function
  – Apply the definition of the second formula for the expected value of a random variable
Workspace
Example 2 For Expected Value

• An Urn is initially filled with 3 red marbles and 4 green marbles
• A marble is selected at random
  – If it is green it is replaced and two additional green marbles are added to the urn
  – If it is red the marble is not replaced
• Next a second marble is drawn at random
• Define a random variable, X, to represent the number of red marbles selected in both draws
• What is its expected value? (Hint: construct a decision tree to determine the probability distribution function of the random variable X)
Workspace
Workspace
Example 3 Roulette

- Players place their bets and the wheel is spun. Assume each number is equally likely to arise on any particular spin.
  - Bet 1: the player wins the amount of the original bet if the wheel lands on a black space (e.g. if I bet $1 and I win, then I will have $2 (including my original $1) and $0 if I lose)
  - Bet 2: the player wins twice the amount of the original bet if the wheel lands on any of the number 1 through 12 (e.g. if I bet $1 and I win, then I will have $3 (including my original $1) and $0 if I lose)
- Consider a gambler that bets $1 on bet 1 and $1 on bet two. What is the gambler’s expected profit?
Workspace
Connection Between the Probability of an Event and Random Variables

• Given probability space \((U,f)\) and event \(E\)
• Define a random variable \(X\)

\[
X(i) = \begin{cases} 
1, & i \in E \\
0, & \text{else} 
\end{cases}
\]

• \(X\) is called an “indicator random variable” for the event \(E\) since its output indicates whether the event \(E\) has occurred or not
Problem

• Show that: $E[X] = P(E)$
Linearity of Expectations

• Given two random variable $X$ and $Y$
• $E[X + Y] = E[X] + E[Y]$
• This holds regardless of any dependencies between the random variables
• It also holds for any number of random variables
  $$E \left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i]$$
• And for multiples of random variables by a constant
  $$E[cX] = cE[X]$$
Why Does this Work?

\[
E[X + Y] = \sum_{i \in U} f(i)(X(i) + Y(i))
\]

Distribute multiplication over addition and use the associative property of addition to split into two sums

\[
= \sum_{i \in U} f(i)X(i) + \sum_{i \in U} f(i)Y(i)
\]

Apply first formula for expected value for each random variable X and Y

\[
= E[X] + E[Y]
\]
Linearity of Expectations Example

• A biased coin (P(H) = 3/5) is flipped 100 times. What is the expected value of the random variable X that represents the number of heads that appear in the sequence?

• Strategy:
  – Define 100 random variables $X_1 \ldots X_{100}$ where the random variable $X_i$ is the indicator random variable that the ith flip came up heads
Example Continued

• What is the distribution function of the random variable $X_i$?

$$f_{X_i}(y) = \begin{cases} 
\frac{3}{5}, & y = 1 \\
\frac{2}{5}, & y = 0 \\
0, & \text{else}
\end{cases}$$

• What is the expected value of this random variable?
Example Continued

• Now we express the random variable \( X \) (the total number of heads) using the individual random variables \( X_1 \ldots X_{100} \)

  (remember, a random variable is just a function defined over the sample space \( U \))

\[
X(y) = \sum_{i=1}^{100} X_i(y)
\]

Applying the expectation operator to both sides and using linearity of expectations we get:

\[
E[X] = E \left[ \sum_{i=1}^{100} X_i \right] = \sum_{i=1}^{100} E[X_i] = \sum_{i=1}^{100} \frac{3}{5} = 60
\]
Linearity of Expectations Example 2

• Consider flipping a fair coin 100 times
• Define the random variable X to be the number of runs of 6 heads observed in the sequence
• What E[X]?
• Hint: Define some indicator random variables to represent that there is a run of heads of length 6 beginning at a particular position in the sequence
• Remember linearity of expectation does not depend on the events being independent
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Linearity of Expectations Example 3: Betting Systems

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Excerpt - Page 1: "... countless betting systems - often arcane, sometimes incomprehensible, always futile - have been
Surprise me! See a random page in this book.

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Martingale Betting

• One of the most popular theories on how to always win at gambling

• Consider a casino game such as roulette where the odds of winning an amount = to the original bet is .49 and the odds of losing the original bet is .51

• Martingale strategy:
  – Bet = $1. If you win: stop. If you lose and total # of bets < k: double bet and try again
Questions

• How much will the player win in the event that she eventually wins a bet before reaching the limit of k bets?
• How much will they lose in the event that they reach the limit of k bets?
Workspace
Questions

• Define a random variable $X$ to represent the players change in money from using the Martingale system

• What is its expected value?
Workspace
Other betting systems

• Many casino games have multiple bets a player can make. Often the outcome of each of the bets are not independent

• Consider a game with two possible bets. The expected value of betting $1 on bet 1 is $u$ (with $u < 0$). The expected value of betting $1 on bet 2 is $v$ (with $v < 0$).

• Show that if a player bets $m$ on bet 1 one and $n$ on bet 2 that the expected value of the combined bet must be less than 0 (you can’t beat the house!)
Workspace
Additional Problems (not covered in lecture)

• Define $X$ to be the number of 1-cycles for a randomly selected permutation (all permutation are equally likely to be selected)

• What is $E[X]$?
Birthday Problem Revisited (not covered in lecture)

• In a class of n people. Let the random variable X be the number of subsets of two people with the same birthday (i.e. how many pairs of people share the same birthday). What is $E[X]$?