CSE 21
Math for Algorithms and Systems Analysis
Lecture 9
Probability
Outline For Today

• Review of multisets and multichoose
• Basic Definition of Probability
• Probability and Set Algebra
• Binomial Probability Distribution
Multisets

- Multisets are unordered collections of elements where repeats are allowed
  - E.g. \{a,a,b,b,c,c,c\}
- The cardinality of a multiset is the total number of elements (including repeats)
- Two multisets A and B are distinct if and only if there exists an element \(x \in A\) with multiplicity \(m\) such that that either \(x \notin B\) or \(x\) appears \(n\) times in \(B\) with \(n \neq m\)
- \{a,b,c,c,a\} = \{c,c,a,b,a\}
- \{a,b,c,c,a\} \neq \{a,b,c,a\}
Representing Multisets with Stars and Bars

• Suppose we are making a size 3 multiset from the elements of a set \( S = \{a, b, c\} \)
• Here are the possible multisets written out with stars and bars

\[
\begin{align*}
***| &= \{a,a,a\} & *|** &= \{a,b,b\} \\
|*** &= \{b,b,b\} & *|** &= \{a,c,c\} \\
|*** | &= \{c,c,c\} & *|*| &= \{a,b,c\} \\
**|* &= \{a,a,b\} & **|* &= \{b,b,c\} \\
**|* &= \{a,a,c\} & |** |* &= \{b,c,c\} \\
**|** &= \{a,a,c\} & |*|** &= \{b,c,c\}
\end{align*}
\]
Multichoose

• We use the notation \( \binom{n}{k} \) to denote the number of size k multisets that can be made from the elements of a n set.

• This is read: “n multichoose k” and can be computed by considering the number of lists that can be made from n-1 bars and k stars

\[
\binom{\binom{n}{k}}{k} = \binom{n - 1 + k}{k}
\]
Typical Multichoose Problems

• There are three restrictions we use with multichoose when allocating a set of n items to two groups A and B (please write the number of allocations such that):
  – Each member of one particular group gets exactly k of the items
  – Each member of one particular group gets at least k of the items
  – Each member of one particular group gets no more than k of the item
Probability

• Probability is really all about counting
• The work we have done up until now will allow us to work with probabilities quite easily
The dots are elements of the universal set $U$. We can think of them as representing many types of elements (e.g. sets, multisets, card hands, words, etc.)
Typical Counting Question

• How many of the elements in U have some property (e.g. how many card hands are straights)?
Related Counting Question

• What if I instead asked what proportion of the elements in U are also in S?

• Answer = \[ \frac{|S|}{|U|} \]

• This notion is very close to our definition of the probability of an event
Basic Definitions for Probability

- U is a set called the sample space
- U represents a set of possible things that can occur
- Additionally we can think of an Event as a subset of the elements of the sample space
- For instance U could be the set of all five card poker hands and the event E could be the set of all 5-card poker hands that are straights.
- If each hand was equally likely to occur we could calculate the probability of the event E as: \[
\frac{|E|}{|U|}
\]
Probability Function

• However, sometimes we wish to encode the notion that not all elements of U are equally likely to occur. We can do this using a probability function

• A probability function is a function from U to the real numbers $[0,1]$

• The function specifies how likely that a particular sample (also called an elementary event) is to occur
Questions About These Definitions

• Let $P$ be a probability function on the set $U$
• What is the value of $\sum_{x \in U} P(x)$

• What do you think the probability of an event $E$ with $(E \subseteq U)$ is?
Counting vs. Probability

- If we define our sample space in such a way that $P(x) = 1 / |U|$ for all elements $x$ in $U$ (i.e. that each elementary event is equally likely to occur then we don’t need any new skills to compute probabilities.
Probability Sample Problems

• Consider drawing a 5-card hand from a deck of cards
• What is your sample space?
• Compute the probability of:
  – Getting at least 2 queens
  – Getting a full house
  – Getting a royal flush
Probability and Counting the Complement

• Suppose we are interested in computing the probability of an event E. How can we compute the probability of this event if we already know the probability of $\overline{E}$?
Probability and the Rule of Sum

• Assume $T_1$, ... $T_m$ are disjoint sets with

\[ T = T_1 \cup T_2 \cdots \cup T_m \]

\[ P(T) = P(T_1) + P(T_2) \cdots + P(T_m) \]
Probability and Inclusion Exclusion

- The rules for sets work pretty much unchanged here (rule of sum is a special case)
- Two events
  \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
- Three events
  \[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \]
  \[ - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \]
Probability and Set Algebra

• Recall the rules we have for set algebra. All of these can be applied to computing probabilities

• Example:

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

• Implies:

\[ P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) \]
Choose your Sample Space Wisely

- My advice is to choose your sample space to be as small as possible while trying to maintain the property that all elementary events are equally likely.
- This allows us to use the proportion of elements with a particular property to compute probabilities.
- Be careful though...
Two methods of counting Computing the Probability of getting two three of a kinds

\[ P(\text{two three of a kinds}) = \frac{\binom{13}{2} \binom{4}{3}^2}{\binom{52}{6}} \]

\[ P(\text{two three of a kinds}) = \frac{\binom{13}{2} \binom{4}{3}^2 \cdot 6!}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47} \]

\[ P(\text{two three of a kinds}) \neq \frac{\binom{13}{2} \binom{4}{3}^2}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47} \]
Probabilities and Allocations

• We allocate 7 green jellybeans to 3 people by selecting someone to receive each jellybean with equal probability

• What is the probability that each person gets at least 1 jelly bean?
Workspace
More Probability Problems

• In the game of 21 a blackjack is when the first two cards dealt to a player consist of a face card (Jack, Queen, or King) or a ten and an ace

• What is the probability of being dealt a blackjack?
Probability and Independence

• Two events $A$ and $B$ are independent if and only if
  \[ P(A \cap B) = P(A)P(B) \]

• Give an example of two events that are independent

• Give an example of two events that are not independent
Binomial Probability Distribution

• Suppose an event has a probability $q$ of occurring on each trial.

• What is the probability of observing exactly $k$ of these events in a sequence of $n$ trials?
More Probability Problems

• A fair coin is flipped 8 times
• What is the probability that it comes up heads exactly 3 times?
More Probability Problems

• A biased coin (probability of heads = .3) is flipped 8 times
• What is the probability that it comes up heads exactly 3 times?
Hypergeometric Probabilities

• Suppose we choose from k marbles a bag containing two types of marbles. There are m red marbles and n green marbles. Each marble of the same color is not distinct.

• The probability that exactly i of the k marbles are red is:

\[
\frac{\binom{m}{i} \binom{n}{k-i}}{\binom{m+n}{k}}
\]

• What about the fact that the marbles of the same color are not distinct. Is there a problem with our solution?
Workspace
• Two baseball teams are competing to win the division. Team A has a record of 90-67. Team B has a record of 87-70. Each team has five games remaining in their season (they do not play each other). Assume the probability of team A winning a particular game is \( \frac{90}{90 + 67} \). The probability of team B winning each game is \( \frac{87}{87 + 70} \).
Problem 1

• What is the probability that team A finishes with more wins than team B?
Workspace
Problem 2

• Suppose team A and B are playing each other for the final five games of the season
• Suppose the probability of team A winning each game is 5/9
• What is the probability that team A will finish the season with more wins than team B?
Workspace
A Game of Probabilities (not part of the class officially but fun!)

- You play a game with two partners. During the game each of you will have either a red hat or a blue hat with equal probability placed on your head.
- You can see the color your partners hat but not of your own
- After viewing the colors of your partners hats each of you retreats to a private room and has the option to either guess the color of your own hat or offer no guess at all
- Your team wins if at least one person guesses correctly and no person guesses incorrectly
- You can come up with a strategy before the game starts but may not communicate during the game
- What is the optimal strategy and what is the probability of your team winning?