CSE 21
Math for Algorithms and Systems Analysis
Lecture 6
Stirling Numbers of the Second Kind
+ Introduction to Functions
Outline For Today

• Redo of Stirling Numbers of the 2\textsuperscript{nd} kind
• Proof by Double Counting
• Intro to Functions
Topic 1: Another try at Stirling Numbers of the Second Kind

- The Stirling number \( n,k \) (written \( S(n,k) \)) gives the number of size \( k \) partitions of an \( n \)-set.
Sample Partitions of a Set (k = 3)
Sample Partitions of a Set (k = 3)
Sample Partitions of a Set (k = 3)
Sample Partitions of a Set (k = 2)
How Can We Derive a Formula for the Stirling number $S(n,k)$?

• What tools do we have in our toolbox?
  – Rule of Product
  – Rule of Sum

• Let’s think about the following procedure for generating all possible partitions of $n$ elements
  – Partition the first $n-1$ elements
  – Modify the each of the partitions in the first step to include the $n$th element
Counting Via Recursion

• Suppose we have a family of sets we are trying to count. Let $f(n)$ be the nth set. Constructing these sets can be accomplished via recursion.

• General procedure
  – Define some base cases (e.g. values of $f(n)$ that can be easily computed)
  – “Pretend” someone has magically given you the values of the function for smaller values of $n$
  – Use these values for smaller values of $n$ to compute the value of the function for $n$
Example: The Powerset

• Let $f(n)$ be the number of subsets of the set $\{1 \ldots n\}$
• $f(1) = \{\{1\}, \emptyset\}$
• $f(n)$ can be constructed from $f(n-1)$ in the following way:
  – Choose an element $x$ in $f(n-1)$ and add it to $f(n)$
  – Choose an element $x$ in $f(n-1)$ and add $x \cup \{n\}$ to $f(n-1)$
• What is $|f(n)|$ in terms of $|f(n-1)|$
How can we derive a formula for the Stirling Number?

- Case A: the nth element is in a set by itself
How can we derive a formula for the Stirling Number?

- Case B: The nth element is not by itself
How can we derive a formula for the Stirling Number?

- The nth element is not by itself
How can we derive a formula for the Stirling Number?

- The nth element is not by itself
Using Rule of Product

• Case A: The nth element occurs in a set with other elements. Choose a partition of n-1 elements of size k, then choose one of the k sets to place the nth element into
• Case B: The nth element occurs in a set by itself. Choose a partition of an n-1 set of size k-1, then add the the nth element as a set by itself
• Therefore the recursion for computing $S(n,k)$ is
  – $S(n,k) = k*S(n-1,k) + S(n-1,k-1)$
  – A closed form formula exists (but you do not have to know it). Later in the class we will learn techniques for converting a recursive formula into a non-recursive one
Topic 2: Proof by Double Counting

- Given an equality LHS = RHS
- Define a set S such $|S|$ can be counted in two different ways such that one of the ways yields LHS and the other RHS
Combinatorial Proofs By Double Counting

• We have already seen this idea several times
  – Example 1:
    \[
    2^n = \sum_{i=0}^{n} \binom{n}{k}
    \]
  – Example 2:
    \[
    \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
    \]
Another Example of Proof by Double Counting

• Prove that: (hint consider a single elimination tournament)

\[ 2^n - 1 = \sum_{i=0}^{n-1} 2^i \]
Challenge Problem for Proof by Double Counting (not covered in class)

- Handshaking Problem
  - Claim: given a room of $n$ people, a sequence of handshakes (all unique, meaning the same two people do not shake hands more than once) takes place. The number of people who shook an odd number of other people’s hands is even.
Topic 3: Introduction to Functions

- An example of a function
Functions Formally

• A function from set $A$ to be $B$ specifies a mapping between each element in the set $A$ and a unique element in $B$

• We write $f : A \rightarrow B$, to indicate that $f$ is a function from $A$ to $B$

• Some terminology
  – $A$ is called the domain of the function
  – $B$ is called the co-domain or the range of the function
Functions Graphically

• How about a function from the numbers 1 through 10 to their parity (even or odd)
Functions in Two Line Form

- Given a domain $S = \{a, b, c, d, e\}$ and a range $T = \{b, f, h\}$ we can write a particular function from $S$ to $T$ in two line form as:

\[
\begin{pmatrix}
a & b & c & d & e \\
b & f & h & h & h
\end{pmatrix}
\]

\[
\begin{align*}
f(a) &= b \\
f(b) &= f \\
f(c) &= h \\
f(d) &= h \\
f(e) &= h
\end{align*}
\]
Operations on Two Line Forms

• Would the following operations change potentially create a new function?
  – Swap the order of the columns
  – Reorder the elements on the top row
  – Reorder the elements on the bottom row
Types of Functions

• Injective – A function is injective if each element in the range is mapped to by no more than one element in the domain.
Types of Functions

• Surjective – a function is surjective if each element in the range is mapped to by at least one element in the domain
Types of Functions

• Bijectons – A function is a bijection if it is both surjective and injective

• Note that a permutation is simply a bijection from a set to itself (more about these tomorrow)
Counting Functions

- No restrictions
Counting Each of these Types of Functions

• Injective
Counting Each of these Types of Functions

• Surjective
Counting Each of these Types of Functions

• Bijective
Applications of Functions

• Each of 6 boys toss a Frisbee to 1 of 4 girls. How many ways can this be done?
Workspace
Challenge Problem (not covered in class)

• A fixed point of a function \( f: S \to S \) is an element \( x \) such that \( f(x) = x \).

• Let \( S = \{1,2,3,4,5,6,7\} \). How many functions from \( S \to S \) are there that have 5 or more fixed points?
Challenge Problem (not covered in class)

• Prove that if the domain and range of a function are both the set $S$ that in order for a function between $S$ and $S$ to be surjective it must also be injective