CSE 21
Math for Algorithms and Systems Analysis
Lecture 4
Multinomial Coefficients and Sets
Quick Review of Last Time

• The number of subsets of size $k$ that can be made from the elements of an $n$-set is

$$\frac{n!}{(n - k)!k!}$$

• Which is also written as $\binom{n}{k}$

• The preceding notation is read as $n$ “choose” $k$. 
Review Problems

• How many Straights (a sequence of 5 ranks in ascending order, Ace can be low or High) can be made from the standard 52 card deck?

• How many Flushes (5 cards of all the same suit) can be made from the standard 52 card deck?
Review Problems Continued

• How many 2 pair poker hands are there (consisting of two pairs with ranks distinct from each other and 1 additional card that is of different rank than either of the two pairs)?
Quick Review of Last Time

• Basics of Decision Trees
  – Decision trees enumerate all particular structure that can be generated via sequential choices
  – In contrast to using the rule of product, the number of choices that can be made at a particular stage can depend on the previous choices constructed
Another Approach to the Bookkeeper Problem

• How many 5 letter words can be made from the letters in the word “MISSISSIPPI” assuming no letter is used more often than it appear in the original word?

• Approach from Last Time
  – Step 1: Write down all letters that appear in the word and the number of times they occur
  – Step 2: Form templates to compose the word based on the multiplicities of each letter in the word
  – Step 3: count the number of words that match each of the templates from step 2
  – Step 4: add up the number of words that match each template
Mississippi

• How many 5 letter words can be made from the letters in the word “MISSISSIPPI” assuming no letter is used more often than it appear in the original word?
Another Approach

• Step 1, and 2 same as before
• Step 3 will be performed differently
• Take a particular template WWXYZ
  – First we select the letters that will appear in the word and their multiplicities
  – Next we compute all possible ways to permute a list of letters with these particular multiplicities
  – Then by the rule of product, the number of words that match the template will be the product of these two quantities
Mississippi Example Continued

• Let’s take a particular template XXWWZ
• How many ways can we select the letters that appear in this word and their multiplicities?

\[ M^1, I^4, S^4, P^2 \]

\[
\binom{3}{2} \binom{2}{1}
\]
Mississippi Example Continued

• Suppose we select the following letters and multiplicities

\[ M^1, S^2, I^2 \]

• How many words (i.e. sequences of letters) can we make using all of the letters above?
Multinomial Coefficients

• Let’s consider one method of computing the number of permutations:
• Choose 1 position for the M to occur
• Choose 2 positions for the S’s to occur
• Choose 2 positions for the I’s to occur

\[
\binom{5}{1} \binom{4}{2} \binom{2}{2} = \frac{5!}{1!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{2!0!} = \frac{5!}{2!2!1!}
\]
Why Does this Formula make Sense?

• Consider that someone gives a list of arrangements of the letters $M^1, S^2, I^2$

• Now someone tells us that we should consider each $S$ distinct from each other and each $I$ distinct from each other

• How could we generate the list of all 5-lists of these letters?
Why Does this Formula Make Sense?

• Choice 1: choose an arrangement of the letters from the list where the I’s and S’s are not considered distinct.

• Choice 2: choose which of the two distinct I’s will go in which position from the original arrangement

• Choice 3: choose which of the two distinct S’s will go in which position from the original arrangement
Why Does this Formula Make Sense?

• Choice 1: n-ways
• Choice 2: 2! Ways
• Choice 3: 2! Ways
• We know from before that there are 5! Ways to arrange these letters when they are all distinct
• Therefore \( n \times 2! \times 2! = 5! \)
• Which implies \( n = \frac{5!}{2!2!} \)
A commission is being formed to recommend cuts to the U.S. budget in exchange for raising the debt ceiling.

The Committee is composed of 3 democrats from the House of Representatives (out of 193), 3 Democrats from the Senate (out of 51), 3 Republicans from the House of Representatives (out of 243) and 3 Republicans (out of 47) from the Senate.

Additionally, there will be roles assigned to some members of the committee (3 people will be on the lunch sub-committee to decide what is eaten each day and 2 people will be in charge of communicating the committees activities to the press each day).

How many committees can be formed?
Multinomial Example Problem
Workspace
Harder Version of Previous Problem

• Suppose the rules for the committee are such that it must be composed of 6 democrats and 6 republicans but the breakdown between the house and senate can be arbitrary. How many committees can be formed in this case?
Workspace
Multinomial Theorem

\[(x_1 + x_2 + \ldots + x_m)^n = \sum_{k_1+k_2+\ldots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}\]

- This looks complicated but we can understand it using similar techniques to what we did for the binomial theorem
Set Identities

- **Associativity**
  \[(P \cap Q) \cap R = P \cap (Q \cap R)\]
  \[(P \cup Q) \cup R = P \cup (Q \cup R)\]

- **Distributivity**
  \[P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)\]
  \[P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)\]

- What relationship do you notice between each pair of identities?
Visualizing Set Relations

\[ P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R) \]
Visualizing Set Relations

\[ Q \cup R \]

\[ P \]

\[ Q \]

\[ R \]
Visualizing Set Relations

\[ P \cap (Q \cup R) \]
Visualizing Set Relations

$P \cap Q$
Visualizing Set Relations

\[ P \quad P \cap R \quad Q \quad R \]
Visualizing Set Relations

\[ P \quad (P \cap Q) \cup (P \cap R) \]

\( P \)
\( Q \)
\( R \)
Other Properties

• Demorgan’s Laws

\[ P \cap Q = \overline{P \cup Q} \]

\[ P \cup Q = \overline{P \cap Q} \]
Show That Demorgan’s Laws are True Graphically
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Applications of Set Algebra Counting the Complement

• Let $U$ be the universal set (that is the set of all elements)

$$\overline{A} = U - A$$

$$|A - B| = |A| - |A \cap B|$$

$$|\overline{A}| = |U| - |U \cap A| = |U| - |A|$$

$$|A| = |U| - |\overline{A}|$$
State Origins

• Out of a class of 32 people how many different combinations of state origins can people have such that at least two people are from the same state? (treat each member of the class as distinct)
Workspace
Applications of Set Algebra (Inclusion Exclusion)

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

• In order to count the size of the union of two sets we can count the size of each set add them and then subtract the intersection
How Many Poker Hands Contain A Straight or a Flush

• A is the set of all straights
• B is the set of all flushes

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\[
\binom{10}{1}^4 \times 4^5 \quad \binom{13}{5}^4 \times 4^4
\]
Applications of Inclusion Exclusion

• How many poker hands have at least 1 pair or 1 three of a kind?
Workspace
Inclusion Exclusion Continued

• Derive a formula for the size of the union of three sets of the form:

\[ |A + B + C| = |A| + |B| + |C| + ??? \]
Workspace