Homework 5

5.1 [10 points]
Derive closed-form expressions for the following recursive formulas:

(a) \( a_n = 2a_{n-1} + 4, \ a_0 = 2 \)
(b) \( a_n = -3a_{n-1} + 3, \ a_0 = 5 \)

(a) By Theorem 8: \( a_n = (6)2^n - 4 \)
(b) By Theorem 8: \( a_n = \frac{17}{4}(-3)^n + \frac{3}{4} \)

5.2 [15 points]
Derive closed-form expressions for each of the following recursive formulas:

(a) \( a_n = 10a_{n-1} - 25a_{n-2}, \ a_0 = 1, a_1 = 2 \)
(b) \( a_n = 5a_{n-1} - 6a_{n-2}, \ a_0 = 2, a_1 = 3 \)

(a) The roots of the characteristic polynomial \( x^2 - 10x + 25 = (x - 5)(x - 5) \) is a double root with \( r = 5 \). The solution will be of the form \( a_n = K_15^n + K_2n5^n \). Solving for \( K_1 \) and \( K_2 \) after plugging in the initial conditions gives: \( a_n = 5^n - \frac{3}{5}n5^n \).

(b) The roots of the characteristic polynomial \( x^2 - 5x + 6 = (x - 3)(x - 2) \) are \( r_1 = 3 \) and \( r_2 = 2 \). Therefore the solution will be of the form \( a_n = K_13^n + K_22^n \). Plugging in the initial conditions gives the values of \( K_1 = -1 \) and \( K_2 = 3 \). Therefore the solutions is \( a_n = (-1)3^n + (3)2^n \).
5.3 [20 points]
In the dice-game of craps a bet on the passline is resolved in the following way. The game consists of repeatedly rolling two 6-sided dice a number of times. The first roll is called the come out roll. If the number comes up 2, 3 or 12 the player immediately loses the passline bet. If the number comes up 7 or 11 the player immediately wins the bet. If the roll is any other number that number is called the point (e.g. the point may be 5). The dice are then repeatedly rolled until either a 7 comes up (in which case the player loses) or the point comes up again (in which case the player wins).

(a) Compute the probability that given a point is established (i.e. the game doesn’t end on the come out roll) that the player will win

(b) Compute the overall probability that the player will win the pass line bet.

(a) We have the following probabilities for each of the possible point values \( P(4) = \frac{3}{36}, P(5) = \frac{4}{36}, P(6) = \frac{5}{36}, P(8) = \frac{5}{36}, P(9) = \frac{4}{36}, P(10) = \frac{3}{36} \). The probability of winning given a particular point is established is for instance 5 is: \( P(win|5) = \frac{4}{36} + \frac{26}{36}P(win|5) \). Therefore \( P(win|5) = \frac{2}{5} \). Analogously we can find the probability of: \( P(win|4) = \frac{1}{4}, P(win|6) = \frac{5}{11}, P(win|8) = \frac{5}{11}, P(win|9) = \frac{2}{5}, P(win|10) = \frac{1}{3} \). Therefore the answer to the problem is:

\[
P(win|point\ established) = \frac{\frac{4}{36} + \frac{2}{5} + \frac{5}{36} + \frac{5}{11} + \frac{5}{11} + \frac{4}{36} + \frac{3}{36}}{1 - \frac{1}{36} - \frac{1}{6} - \frac{1}{36} - \frac{1}{6} - \frac{1}{36}} = 0.4061
\]

(b) \( p(win) = p(point\ established)p(win|point\ established) + p(win\ on\ comeout) = \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{5}{11} + \frac{5}{11} + \frac{4}{36} + \frac{3}{36} + \left( \frac{1}{6} + \frac{2}{36} \right) = 0.4929 \)

5.4 [15 points]
How many sequences of binary digits of length \( n \) can be made such that the sequence ‘10’ never occurs?

We can solve this using a first-order linear recurrence. To construct a sequence of size \( n \) we either start with a 0 and have \( f(n - 1) \) ways to complete the sequence or we start with a 1 in which case we have only 1 way to finish the sequence: Therefore, \( f(n) = f(n - 1) + 1 \).
\[ f(n) = n + 1 \]

5.5 [15 points]
A casino game allows the player to win his or her original bet with probability .49 and lose his or her bet with .51 probability. Suppose a player implements the following betting strategy. Start by betting $1. If win, then quit. If lose and total number of bets so far is less than \( n \), then double the bet, otherwise quit.

(a) Show that the probability of the player winning $1 using this strategy approaches 1 as the limit on the number of bets gets higher and higher.

(b) Provide a formula for the expected value of this betting strategy as a function of the bet limit \( n \). Do you think this is a good betting strategy?

(a) The probability of the player winning can be defined recursively as follows: \( P(n) = .49 + .51P(n-1) \) with \( P(0) = 0 \). This is a linear first order recurrence and can be solved with Theorem 8. Therefore \( P(n) = 1 - .51^n \). As \( n \) gets large we can see that this converges to 0.

(b) We can write the expected value recursively as \( E(n) = .49 - .51 + .51(2)E(n-1) \). \( E(0) = 0 \). By Theorem 8: \( E(n) = (-1)1.02^n + 1 \). While the probability of winning goes up to 1 asymptotically the expected value goes to negative infinity, so I would so it is not a good betting strategy.

5.6 [10 points]
How many unique ways can a strip of squares of size \( n \times 1 \) be tiled using the following types of tiles:

1. Red 1 square by 1 square tiles
2. Blue 1 square by 1 square tiles
3. Green 1 square by 1 square tiles
4. Black 1 square by 2 square tiles

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This can be written as a second-order linear recurrence: $f(n) = 3f(n - 1) + f(n - 2), f(1) = 3, f(2) = 10$. The characteristic polynomial is $x^2 - 3x - 1$. The two roots are: $r_1 = \frac{3 + \sqrt{13}}{2}$ and $r_2 = \frac{3 - \sqrt{13}}{2}$. Therefore the solution is of the form $f(n) = K_1 r_1^n + K_2 r_2^n$. Plugging in the initial values and solving for $K_1$ and $K_2$ numerically we get:

$$f(n) = .9160 \left(\frac{3 + \sqrt{13}}{2}\right)^n + .840 \left(\frac{3 - \sqrt{13}}{2}\right)^n$$