Homework 4

4.1 [15 points]
Team A and Team B are competing to have the best record in their conference. Team A’s record is 20-10 while Team B’s is 18-12. Each team has 4 games left in its season. Assuming that team A and B are not playing each other and that the probability of each team winning each of its last four games is $\frac{3}{5}$ (with the outcome of all games independent), what is the probability that Team B will finish with a better record than team A?

Team finish with a better record if team B goes 4-0 and team A goes 1-3 or 0-4. Or if Team B goes 3-1 and team A goes 0-4.

Answer =

\[
\binom{4}{4} \left( \frac{3}{5} \right)^4 \left( \binom{4}{0} \left( \frac{2}{5} \right)^4 + \binom{4}{1} \left( \frac{3}{5} \right) \left( \frac{2}{5} \right)^3 \right) + \binom{4}{1} \left( \frac{3}{5} \right)^3 \left( \frac{2}{5} \right) \binom{4}{0} \left( \frac{2}{5} \right)^4
\]

\[\blacksquare\]

4.2 [10 points]
Let X be a random variable representing the number of Clubs in a 10 card poker hand and Y be a random variable representing the number of Diamonds. What is E[X + Y]?

We can define 10 indicator random variables $X_1 \ldots X_{10}$ where $X_i$ takes on value 1 iff the ith card dealt is a diamond. Analogously we define similar random variables for Y as $Y_1 \ldots Y_{10}$. By construction $X = \sum_{i=1}^{10} X_i$ and $Y = \sum_{i=1}^{10} Y_i$. By linearity of expectations:

Answer is $E[X + Y] = \sum_{i=1}^{10} E[X_i] + \sum_{i=1}^{10} E[Y_i] = 10 \left( \frac{1}{4} \right) + 10 \left( \frac{1}{4} \right) = 5$.

\[\blacksquare\]

4.3 [10 points]
4 boys each select independently and uniformly at random one of 3 girls to throw a frisbee to. Let X represent the number of frisbees that the first girl gets. What is E[X]? What is Var[X]?

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We define four indicator random variables $X_1 \ldots X_4$ that represent the event that the $i$th boy throws his frisbee to the first girl. By construction $X = X_1 + X_2 + X_3 + X_4$. By linearity of expectations:

$$E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = 4 \times \frac{1}{6} = \frac{2}{3}$$


4.4 [15 points]
Given an image of a face, you observe that you can see the teeth of the person in the image (TS = teeth showing) and that their brows are furrowed (BF = brows furrowed). Assuming that there are only two types of facial expressions joyful and angry and that: $P($joyful$) = .6$, $P($angry$) = .4$, $p(TS|angry) = .5$, $P(BF|angry) = .6$, $P(TS|joyful) = .8$, and $P(BF|joyful) = .05$.

Additionally assume that TS and BF are conditionally independent given the expression (angry or joyful). What is the probability that the image of the face in the picture you see is of someone who is angry? In other words, what is $P($angry$|BF \cap TS)$?

Hint: by Bayes’ rule: $P($angry$|BF \cap TS) = \frac{P(BF \cap TS|angry)P($angry$)}{P(BF \cap TS)}$.

$$P($angry$|BF \cap TS) = \frac{P(BF \cap TS|angry)P($angry$)}{P(BF \cap TS)}$$

$$= \frac{P(BF \cap TS|angry)P($angry$)}{P($angry$)P(BF \cap TS|angry) + P($joyful$)P(BF \cap TS|joyful)}$$

$$= \frac{(.6)(.5)(.4)}{(.6)(.5)(.4) + (.8)(.05)(.6)} = \frac{5}{6}$$


4.5 [15 points]
Derive closed form expressions for the following recursive formulas.

(a) $a_n = 5a_{n-1} - 12, n \geq 0, a_0 = 3$

(b) $b_n = 7b_{n-1} - 1, n \geq 0, b_0 = 7$

(a) Simply apply Theorem 8 from the book: $a_n = 3$

(b) Simply apply Theorem 8 from the book: $b_n = (6 + 5/6)7^n + \frac{1}{6}$
4.6 [15 points]
Derive closed form expressions for the following recursive formulas

(a) \( a_n = 4a_{n-1} + 12a_{n-2}, n \geq 0, a_0 = 3, a_1 = 5 \)

(b) \( b_n = 6b_{n-1} - 9b_{n-2}, n \geq 0, b_0 = 7, b_1 = 2 \)

(a) Simply apply Theorem 9. The characteristic polynomial is \( 0 = x^2 - 4x - 12 \). Therefore, \( r_1 = -2 \) and \( r_2 = 6 \). The solution will have the form \( a_n = C_1(-2)^n + C_26^n \). Plugging in the initial values and solving for \( C_1 \) and \( C_2 \) yields: \( a_n = \frac{13}{8}(-2)^n + \frac{11}{8}6^n \)

(b) Simply apply Theorem 9. The characteristic polynomial is \( 0 = x^2 - 6x + 9 \). Therefore, we have a repeated root with \( r = 3 \). The solution will have the form \( b_n = C_13^n + C_2n3^n \). Substituting in \( n = 0 \), we immediately obtain \( C_1 = 7 \). Substituting in \( n = 1 \) we get \( 2 = 7(3) + 3C_2 \). Therefore \( C_2 = -\frac{19}{3} \).
Answer \( b_n = 7 \times 3^n - \frac{19}{3}n3^n \).

4.7 [15 points]
Prove using Induction that for all natural numbers \( n \):

\[
\sum_{i=1}^{n} i^3 = \left( \sum_{i=1}^{n} i \right)^2
\]

Recall that: \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)

Base case \( i = 1 \): \( 1^3 = 1^2 \).
Inductive step: assume $\sum_{i=1}^{n-1} i^3 = \left( \sum_{i=1}^{n-1} i \right)^2$.

\begin{align*}
\sum_{i=1}^{n} i^3 & = n^3 + \sum_{i=1}^{n-1} i^3 \\
& = n^3 + \left( \sum_{i=1}^{n-1} i \right)^2 \\
& = n^3 + \left( \sum_{i=1}^{n} i - n \right)^2 \\
& = n^3 + \left( \sum_{i=1}^{n} i \right)^2 - 2n \sum_{i=1}^{n-1} i + n^2 \\
& = \left( \sum_{i=1}^{n} i \right)^2 + n^3 - 2n \frac{n^2 + n}{2} + n^2 \\
& = \left( \sum_{i=1}^{n} i \right)^2 + n^3 - n^3 - n^2 + n^2 \\
& = \left( \sum_{i=1}^{n} i \right)^2 \quad (1)
\end{align*}