Homework 1

1.1 [15 points]
An Illinois License Plate contains a letter A-Z, followed by 6 digits 0-9.

(a) How many Illinois License Plates contain exactly 1 B?
(b) How many Illinois License Plates contain exactly 1 2?
(c) How many Illinois License Plates contain no B’s or 2’s?

Part (a) the B must be the first and only letter: $10^6$
Part (b) the 2 can appear as any of the digits: $26 \times 6 \times 9^5$
Part (c) $25 \times 9^6$

1.2 [15 points]
How many different ways can 2 red chairs, 1 blue chair, and 2 green chairs be placed in a row if:

(a) If no two chairs next to each other are the same color?
(b) All of the chairs of each color are together?
(c) Each chair has one neighbor of a different color?

Part (a): We can construct a decision tree to enumerate all possible arrangements. For space considerations we will break it down into three different trees depending on the first chair:
   First chair Red:
First chair Green:

First chair Blue:
By counting the leaf nodes in the decision tree that match the criterion of the problem we obtain the solution of 12.

Part (b): We do not need a decision tree for this problem. The answer is 3! (once we arrange the colors R,G,B in order we have the position of all 5 chairs determined)

Part (c): This is everything except the cases where we begin or end with two of the same color. Total arrangements is \( \binom{5}{2,2,1} \). Subtract off cases where we begin or end with 2 red or 2 green (obtained using the decision tree from part (a)): RRBGG, RRRGB, RRGGB, GGRBB, GGRBR, GGBRR, BRRGG, RBRGG, BGGRR, GBGRR. Answer is \( \binom{5}{2,2,1} - 10 = 20 \)

1.3 [10 points]
How many words of length 5 with all letters distinct can be made from the letters in the word “SEQUOIA”?

There are 7 unique letters, therefore the answer is \( \frac{7!}{2!} = 2520 \)

1.4 [15 points]
How many distinct words of length 4 can be made from the letters in the word BOOKKEEPER assuming that no letter can be used more often than it appears in the word “BOOKKEEPER”?

Let’s count the occurrence of each letter = B^1, O^2, K^2, E^3, P^1, R^1.

Possible templates:

\[
\text{XXXY} = \binom{1}{1} \times \binom{5}{1} \times \binom{4}{1} \\
\text{XXYY} = \binom{3}{2} \times \binom{4}{2,2}
\]
\[ XXYZ = \binom{3}{1} \times \binom{5}{2} \times \binom{4}{2,1,1} \]
\[ WXYZ = \binom{6}{4} \times \binom{4}{1,1,1,1} \]
Total = 758

1.5 [15 points]
A snippet of DNA is composed of sequences of nucleotides (A,C,T,G). All variants of a particular gene fragment contain exactly 6 nucleotides. Additionally:

1. The nucleotide A appears no more than twice
2. The sequence does not begin or end with a C or a G
3. The 3rd nucleotide is a T

How many variants of this particular gene fragment are there?

We can solve the problem by considering 3 cases and using the rule of product to count each of them.

Case 1: A does not appear in the middle 4 positions
Choose first and last nucleotide: \{AA, AT, TA, TT\} = 4 choices
Choose middle 4 nucleotides (1 position is already fixed to be T): \(3^3\) choices

Case 2: A appears once in the middle 4 positions
Choose first and last nucleotide: \{AT, TA, TT\} = 3 choices
Choose position for the A in the middle 4 = 3 choices
Choose the remaining 3 nucleotides in the middle 4 (1 position is already fixed to be T) = \(3^2\)

Case 3: A appears twice in the middle 4 positions
Choose first and last nucleotide: \{TT\} = 1 choice
Choose positions for the A in the middle 4 = 3 choices
Choose remaining positions in the middle 4 (1 position is already fixed to be T) = 3 choices

Answer = \(4 \times 3^3 + 3 \times 3 \times 3^2 + 3 \times 3 = 198\)
A California License plate contains a digit 0-9 followed by 3 letters A-Z and then 3 digits 0-9. If we consider the lexicographical order on the set of all license plates defined by the cartesian product: \( \{0 \ldots 9\} \times \{A \ldots Z\} \times \{A \ldots Z\} \times \{A \ldots Z\} \times \{0 \ldots 9\} \times \{0 \ldots 9\} \times \{0 \ldots 9\} \) which license plate comes 1052 before 1HEY588?

changing Y to an X brings us 1000 before 1HEY588. Then we simply need to subtract off 52 from 88 to get 36. Therefore the answer is 1HEX536 ■