8.1

5. $s_0 = 1, s_1 = 1, s_2 = 1 + 2 \times 1 = 3, s_3 = 3 + 2 \times 1 = 6, s_4 = 6 + 2 \times 3 = 12.$

22. (a) The recurrence in words is, we’ll have exactly the same number of rabbits as last month plus 4 times the number of rabbits we had 2 months ago. This gives $r_n = r_{n-1} + 4r_{n-2}.$ (b) $r_0 = 1, r_1 = 1, r_2 = 5, r_3 = 9, r_4 = 29, r_5 = 65, r_6 = 181.$ (c) If we keep doing this calculation, we get that $r_{12} = 49661.$

36. (a) The empty string, $\varepsilon$, is the zero length string, then 0, 1, 00, 01, 10, 11, 000, 010, 011, 100, 110, 0000, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101 are the rest. (b) From (a) we have $d_0 = 1, d_1 = 2, d_2 = 4, d_3 = 7, d_4 = 13.$ (c) We can break it into two cases. If the first (leftmost) bit of the string is a 0, we can have any length $(n-1)$-string follow it. For the other case, if the string starts with a 1, it can either start with 10____ and then be followed by any $(n-2)$-string or it can start with 110____ and then be followed by any $(n-3)$ length string. This gives the recurrence $d_n = d_{n-1} + d_{n-2} + d_{n-3}.$ (d) We can solve for $d_5 = 13 + 7 + 4 = 24.$

53. (a) $g_3 = 1$ since the only way he can play 3 games is to LLL. Similarly $g_4 = 1$ since the only way to play 4 games is to WLLL. Finally $g_5 = 2$ for LWLL and WLLL. (b) $g_6 = 4$ for WWWLLL, WLLL, LLWLLL, LWLLL (c) This is the same type of argument as problem 36. We can either start with a W and then have any $(n-1)$ length game afterwards, or we can start with a LW____ or we can start with a LLW____. This gives the recurrence $g_n = g_{n-1} + g_{n-2} + g_{n-3}.$ This checks for $g_6.$

8.2

5. Substituting we get $c_k = 3c_{k-1} + 1 = 3[3c_{k-2} + 1] + 1 = 3[3[3c_{k-3} + 1] + 1] + 1 = \cdots = 3^i c_{k-i} + \sum_{j=0}^{i-1} 3^j.$ Setting $i = k - 1$ gives $c_k = 3^{k-1} c_1 + \sum_{j=0}^{k-2} 3^j = 3^{k-1} + \frac{3^{k-1} - 1}{2} = \frac{3^{k-1} + 1}{2} = \frac{3^k - 1}{2}.$
12. Substituting gives $s_k = s_{k-1} + 2k = s_{k-2} + 2(k - 1) + 2k = s_{k-3} + 2(k - 2) + 2(k - 1) + 2k = \cdots = s_0 + 2 \sum_{j=0}^{k} j = 3 + 2(k + 1)(k - 2) \sum_{j=0}^{k} j = 3 + 2(k + 1)k - 2(k + 1) = 3 + k(k + 1)$.

20. On day 1 she runs it in 2:57, day 2 2:54, day 3 2:51, ..., day 14 2:18.

43. Assuming $k$ is even, doing substitution gives

\[ a_k = a_{k-2} = a_{k-4} = \cdots = a_1 = a_0 = 2 \]

and if $k$ is odd we get

\[ a_k = \frac{a_0}{2a_0 - 1} \]

in other words, the series alternates between 2 and 2/3. The explicit formula is

\[ a_k = \begin{cases} 2 & \text{if } k \text{ even} \\ 2/3 & \text{if } k \text{ odd} \end{cases} \]

We can verify this by induction. We want to show that the formula above for $a_k$ is true for all $k$. We can verify the two base cases $a_0, a_1$ very easily.

Now the induction hypothesis is that given the formula above is true for all $0 \leq i < k$ then the property is true for $k$. We know $a_k = \frac{a_{k-2}}{2a_{k-2} - 1}$ so

\[ a_k = \begin{cases} 2/3 & \text{if } k - 1 \text{ is even} \\ 2 & \text{if } k - 1 \text{ is odd} \end{cases} \]

which is exactly equivalent to the original formula we set out to prove.

3. 8.3

2. b. and f. are.

4. (a) We must solve the simultaneous equations $0 = C + D$ and $5 = 3C - 2D$ which gives $C = 1, D = -1$. $b_2 = 9 - 4 = 5$. (b) We must solve the simultaneous equations $3 = C + D$ and $4 = 3C - 2D$ which gives $C = 2, D = 1$. $b_2 = 2 \cdot 3^2 + (-2)^2 = 22$.

8. (a) For $k > 1$ we get a recurrence of the form $t^k = 2t^{k-1} + 3t^{k-2}$ which is equivalent to $t^2 = 2t + 3$ or $t^2 - 2t - 3 = 0$. Factoring gives $(t-3)(t+1) = 0$ implying the roots are 3 and -1, so any $t$ of the form $t = A \cdot 3^n + B \cdot (-1)^n$ satisfies the recurrence relation. (b) Assuming the solution is of the form $a_n = A3^n + B(-1)^n$, to solve for $A, B$ we must solve the simultaneous equations $a_0 = 1 = A + B$ and $a_1 = 2 = 3A - B$. After solving we get the final answer of $a_n = (3/4)3^n + (1/4)(-1)^n$. 

2
13. The characteristic equation is $t^2 - 2t + 1 = 0$ which factors to $(t - 1)^2 = 0$ which has a single root of 1. So we know the solution must be of the form $r_k = A1^n + Bn1^n = A + Bn$. Solving the simultaneous equations $r_0 = 1 = A$ and $r_1 = 4 = A + B$, gives a final solution of $r_n = 1 + 3n$.

15. The characteristic equation is $t^2 - 6t + 9 = 0$ which factors to $(t - 3)^2 = 0$ having a single root at 3. So we know the solution must be of the form $t_k = A3^n + Bn3^n$. Solving the simultaneous equations $t_0 = 1 = A$ and $t_1 = 3 = 3A + 3B$ gives a solution of $t_k = 3^n$.

25. We can write the characteristic equation as $(1/6)r^2 - r + (5/6) = 0$ and multiplying through by 6 and factoring gives $(r - 5)(r - 1) = 0$ so the roots are 5 and 1. We know the solution is of the form $p_n = A5^n + B$ and solving the simultaneous equations $p_0 = 1 = A + B$ and $p_{300} = 0 = A5^{300} + B$ gives

$$p_n = (1 - \frac{5^{300}}{5^{300} - 1})5^n + \frac{5^{300}}{5^{300} - 1}$$

Solving (in MATLAB) for $p_{20} = 1$ (to computer precision).