6.6

1. \[ \binom{n}{0} = \frac{n!}{0!(n-0)!} = 1 \]

9. \[ \binom{6}{4} = \frac{5}{4} + \frac{5}{3} = 5 + 10 = 15 \]
   \[ \binom{6}{5} = \frac{5}{5} + \frac{5}{4} = 1 + 5 = 6 \]

12. \[ \binom{n+3}{r} = \binom{n+2}{r-1} + \binom{n+2}{r} \]
   \[ = \binom{n+1}{r-2} + 2\binom{n+1}{r-1} + \binom{n+1}{r} \]
   \[ = \binom{n}{r-3} + \binom{n}{r-2} + 2\binom{n}{r-2} + \binom{n}{r-1} + \binom{n}{r} \]
   \[ = \binom{n}{r-3} + 3\binom{n}{r-2} + 3\binom{n}{r-1} + \binom{n}{r} \]

19. \[ \binom{2n}{n} = \binom{n+n}{n} = \binom{n}{0} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n} \]
   \[ = \binom{n}{0} + \binom{n}{1}\binom{n}{1} + \cdots + \binom{n}{n-1}\binom{n}{n-1} + \binom{n}{n} \]
   \[ = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2. \]

- **Claim:** \( \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \)
  **Proof:** One way to think of \( \binom{n}{k} \) is to imagine counting the number of ways to get exactly \( k \) heads in a series of \( n \) tosses of a coin. There are \( k \) positions in the series to be chosen for the heads, and the rest must be tails, so there are \( \binom{n}{k} \) different ways to select the positions for the heads. Note that no two choices of \( k \) can result in the same sequence of coin tosses. If we then add up over all choices of \( k \), we correctly count all possible sequences of \( n \) coin tosses, which is exactly \( 2^n \).

- **Claim:** \( \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r} \).
  **Proof:** Imagine that we have two disjoint sets: \( A \) of size \( m \) and \( B \) of size \( n \), and we wish to select \( r \) elements from \( A \cup B \) (where \( r \leq m \) and \( r \leq n \)). Clearly, there are \( \binom{m+n}{r} \) ways to do this. We can also count this by first choosing a number \( k \) of elements to pick from set \( A \), which then forces \( r-k \) elements to be chosen from set \( B \). Since no two choices for \( k \) can result in the same final selection, we can use the addition rule to add up the counts from all possible choices of \( k \). This results in the the left-hand side of the equation:

\[ \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r} \]