6.1

5. Let \( E \) be the event that the chosen card is at least 10. Written as a set,
\[
E = \{10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit, A\spadesuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit\}.
\]
The set has size \(|E| = 5 \times 4 = 20\), so the probability is 20/52.

14. Since each person can be in one of two states (ill or not ill), and there are three people, the sample space contains \(2^3 = 8\) equally probable outcomes: \( S = \{III, IIN, INI, INN, NII, NIN, NNI, NNN\} \). For example, \( INI \) represents the outcome where persons 1 and 3 become ill, but person 2 does not.

(a) Let \( E_1 \) be the event that exactly one person is ill. There are only 3 possible ways this can happen, so \( E_1 = \{INN, NIN, NNI\} \). The probability of this event is \(|E_1|/|S| = 3/8\).

(b) Let \( E_2 \) be the event that at least two people are ill. If we write down every way this can happen, we get \( E_2 = \{IIN, INI, NII, III\} \). The probability of this event is \(|E_2|/|S| = 4/8 = 1/2\).

(c) Let \( E_3 \) be the event where nobody becomes ill. There is only one such outcome, \( NNN \). So the probability of this event is 1/8.

18. (a) Let \( E_1 \) be the event where two blue balls are drawn: \( E_1 = \{B_1B_1, B_1B_2, B_2B_1, B_2B_2\} \). \(|E_1| = 4\), and the probability of the event is 4/9.

(b) Let \( E_2 \) be the event where the two balls are different colors: \( E_2 = \{WB_1, WB_2, B_1W, B_2W\} \). The probability of this event is \(|E_2|/|S| = 4/9\).

21. (a) Another way to view this question is to ask how many multiples of 3 are positive two-digit integers. The smallest possible multiple is 3 \(\times\) 4 = 12, and the largest is 3 \(\times\) 33 = 99. Everything in between 4 and 33 also qualifies, so subtracting the lower limit from the upper limit and adding one, there are 33 − 4 + 1 = 30 such multiples.

(b) The sample space is all positive two-digit numbers, so \( S = \{10, 11, \ldots, 99\} \) and \(|S| = 99 − 10 + 1 = 90\). The probability of randomly choosing a multiple of 3 is 30/90 = 1/3.

24. (a) If \( n \) is even, then \( |n/2| = n/2 \) is an integer, so the subarray has \( n/2 − 1 + 1 = n/2 \) elements. If \( n \) is odd, then the subarray has \( |n/2| = (n − 1)/2 \) elements.

(b) The sample space here is the set of indices into the array, ie. \( \{1, 2, \ldots, n\} \). The event of interest is \( E = \{1, 2, \ldots, |n/2|\} \). If \( n \) is even, then the probability is \(|E|/|S| = (n/2)/n = 1/2\). If \( n \) is odd, then the probability is \((n − 1)/2n\).

31. This problem is similar to #24. We simply have to find the smallest and largest multiples of 3 that fall into the range 1 to 1001, which turns out to be 1 and 333 respectively. There are 333 − 1 + 1 multiples.

6.2

3. If team A wins 4 games in a row, we can enumerate all the possible outcomes where A wins as follows:
\[
S = \{AAAA, BAAAA, BBAAAA, BBAAAAA\},
\]
and \(|S| = 4\).

9. (a) We can first choose among 3 ways to get from A to B, and then choose from 5 ways to get from B to C, resulting in \(3 \times 5 = 15\) possible routes.

(b) Since we are free to re-use roads on the return trip, we have 3 choices for the first road, 5 for the second, 5 again for the third, and 3 again for the fourth, for a total of \(3 \times 5 \times 3 = 225\) possible routes.

(c) Now we cannot use the same road twice in the round trip. We still have \(3 \times 5\) ways to get from A to C, but on the return trip, there are only 4 ways back from C to B, and 2 ways from B to A. This results in \(3 \times 5 \times 4 \times 2 = 120\) possible routes.
11. (a) We can represent the string as a sequence of blank spaces to be filled: \[ \_ \_ \_ \_ \_ \_ \_ \_ \_ \] where each space has two possible symbols to fill it. There are 8 spaces, and two choices (0 or 1) for each space, resulting in \( 2^8 = 256 \) possible strings.

(b) Now we can restrict the choice for the first three spaces to only one symbol (0), so there are \( 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32 \) possible strings.

(c) Similarly, we can restrict the choice for the first and last symbols, resulting in \( 2^6 = 64 \) strings.

(d) Since there are 256 unique 8-bit strings, each one of these can be assigned to an EBCDIC symbol.

14. (a) There are 26 choices for each of the first four symbols and 10 choices for each of the last three symbols. This results in
\[
26 \times 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^4 \times 10^3
\]
possible license plates.

(b) If we restrict the first symbol to \( A \) and the last symbol to 0, there are \( 1 \times 26 \times 26 \times 26 \times 10 \times 10 \times 1 = 26^3 \times 10^2 \) such license plates.

(c) Restricting all of the first four symbols to \( TGIF \) results in
\[
1 \times 1 \times 1 \times 1 \times 10 \times 10 \times 10 = 10^3
\]
plates.

(d) We are free to choose any of the 26 letters for the first symbol, but after that we can only pick from the remaining 25 for the second symbol, then there are 24 left for the third and 23 for the fourth. Similarly, we can choose from 10 symbols for the first digit, then 9 for the second and 8 for the third. We then get \( 26 \times 25 \times 24 \times 23 \times 10 \times 9 \times 8 \) possible license plates with all distinct symbols.

(e) If we fix the first two symbols to \( AB \), we get \( 1 \times 1 \times 24 \times 23 \times 10 \times 9 \times 8 \) distinct license plates.

17. There are 14 possible ways to choose the officers, as illustrated on page A-46 in the text. The illustrated decision tree has either 2 or 3 branches at step 2, depending on who was chosen in step 1, so the multiplication rule cannot be used. The decisions cannot be reordered to make the multiplication rule possible.

18. (a) The calculation for step 4 is incorrect because it may be the case that none of the first three symbols are digits. If this is the case, then there are still 10 choices for the last symbol, not 7.

(b) There are still 23 choices for the first symbol. There are 10 choices for the last symbol. We have now used 2 of the 36 available symbols, so there are 34 choices for the second symbol and 33 for the third. The total number of PINs satisfying the given conditions is \( 23 \times 34 \times 33 \times 10 \).

24. The outer loop runs 50 – 5 + 1 = 46 times. Each step through the outer loop runs the inner loop 20 – 10 + 1 = 11 times. The code in the inner loop runs a total of 46 \times 11 \text{ times}.

26. It helps to first break down the problem into cases based on the length of the string: 1 digit, 2 digits, 3 digits, 4 digits or 5 digits. Since we are only interested in numbers containing a 2, 3, 4 and a 5, none of the numbers represented by 3 or fewer digits are of interest.

In the case where the number has 4 digits, we know that each digit must come from the set \{2, 3, 4, 5\}, so all that is left is to count the permutations of this set: \( 4! \).

When the number has 5 digits, we know that four of them come from \{2, 3, 4, 5\} and the remaining digit can be any of \{0, 1, 6, 7, 8, 9\}. However, if the first digit in the string is not from \{2, 3, 4, 5\}, it also cannot be a 0, so we should treat this case separately. If the last four digits all come from \{2, 3, 4, 5\}, then the first digit must come from \{1, 6, 7, 8, 9\}. There are \( 5 \times 4! \) such numbers.

Otherwise, we can simply take the number of permutations of \{2, 3, 4, 5\}, then choose a position to insert the fifth digit (between the first and second, second and third, third and fourth, or last), and multiply by the number of possible symbols to put in that position. This results in \( 4! \times 4 \times 6 \) possible numbers.

Adding all of these cases together results in \( 4! + 5 \times 4! + 4! \times 4 \times 6 \).

29. (a) This is the number of permutations of 9 symbols: \( 9! \).

(b) We can treat \( AL \) as if it was a letter, in which case there are only 8 letters to permute: \( 8! \).

(c) Similarly, if we treat \( GOR \) as a letter, there are only 7 to permute: \( 7! \).

30. (a) This is the number of permutations of 6 items: \( 6! \).

(b) If we fix the first (aisle) seat to be the doctor, there are 5 people left to permute: \( 5! \).

(c) We can merge each of the couples together and then permute the 3 couples: \( 3! \).

36. (a) There are 9 letters total. We can choose any of them for the first symbol, then one of the remaining 8 for the second, and one of the remaining 7 for the third: \( 9 \times 8 \times 7 = 504 \).

(b) \( 9 \times 8 \times 7 \times 6 \times 5 \times 4 \).

(c) Fixing the first letter to \( A \), we get \( 1 \times 8 \times 7 \times 6 \times 5 \times 4 \).

(d) Fixing the first two letters to be \( OR \), we get \( 1 \times 1 \times 7 \times 6 \times 5 \times 4 \).
6. (a) One way to solve this is to ignore the case of the blank plate and then subtract it out at the end. This way, we can treat the letter and number sections independently and multiply them together.

There are between 0 and 3 letters, resulting in \(26^0 + 26^1 + 26^2 + 26^3\) possible letter strings. Similarly, there are between 0 and 4 numbers, resulting in \(10^0 + 10^1 + 10^2 + 10^3 + 10^4\) number strings. We can multiply these together and subtract out the blank plate case: \((26^0 + 26^1 + 26^2 + 26^3)(10^0 + 10^1 + 10^2 + 10^3 + 10^4) - 1\).

(b) Using the same setup as part (a), we can subtract out the offending letter combinations as follows:

\[
(26^0 + 26^1 + 26^2 + 26^3 - 85)(10^0 + 10^1 + 10^2 + 10^3 + 10^4) - 1.
\]

This assumes that the empty letter arrangement is not one of the offending 85.

7. We can first consider the case where the number displayed is an integer. Since there are 8 digits, this allows for 99999999 positive integers, 99999999 negative integers, and 0, for a total of 199999999 integers.

For non-integers, we can avoid over-counting for cases like 1.9, 1.90, 1.900, by assuming that the right-most digit is non-zero.

If there are three digits displayed, the decimal can be either ".xxx", "x.xx" or "xx.x". This results in \(9 \times 10 \times 10\) three-digit non-integers.

If there are two digits displayed, the decimal point can be either in between or to the left of the digits. If it’s in between, there are 9 choices for the right symbol and 9 choices for the left (no leading or trailing 0s). If the decimal is to the left of both digits, there are 9 choices for the right and 10 choices for the left. So there are \(9^2 \times 10 \times 10\) two-digit non-integers.

If there are three digits displayed, the decimal can be either ".xxx", "x.xx" or "xx.x". This results in \(10 \times 10 \times 9 \times 10 \times 9 + 9 \times 10 \times 9\) numbers.

In general, for a \(k\)-digit non-integer, there will be \(10^{k-1} \times 9 + (k - 1) \times 9 \times 10^{k-2} \times 9\) possible numbers, where the first term represents the case where the decimal leads, and the second term represents all other cases where the decimal is between digits. Since \(k\) can vary from 1 to 8, we get

\[
9 + (90 + 81) + (900 + 810 \times 2) + (9000 + 8100 \times 3) + (90000 + 81000 \times 4) + (900000 + 810000 \times 5) + (9000000 + 8100000 \times 6) + (90000000 + 81000000 \times 7).
\]

Multiplying by 2 to account for the negative numbers gives us 1440000000 non-integers. Adding this to the number of integers results in 1639999999 possible numbers.

10. (a) One way to solve this is to ignore the case of the blank plate and then subtract it out at the end. This way, we can treat the letter and number sections independently and multiply them together.

There are between 0 and 3 letters, resulting in \(26^0 + 26^1 + 26^2 + 26^3\) possible letter strings. Similarly, there are between 0 and 4 numbers, resulting in \(10^0 + 10^1 + 10^2 + 10^3 + 10^4\) number strings. We can multiply these together and subtract out the blank plate case: \((26^0 + 26^1 + 26^2 + 26^3)(10^0 + 10^1 + 10^2 + 10^3 + 10^4) - 1\).

(b) Using the same setup as part (a), we can subtract out the offending letter combinations as follows:

\[
(26^0 + 26^1 + 26^2 + 26^3 - 85)(10^0 + 10^1 + 10^2 + 10^3 + 10^4) - 1.
\]

This assumes that the empty letter arrangement is not one of the offending 85.

14. (a) One way to solve this is to ignore the case of the blank plate and then subtract it out at the end. This way, we can treat the letter and number sections independently and multiply them together.

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(b) Using the same setup as part (a), we can subtract out the offending letter combinations as follows:

\[
(26^0 + 26^1 + 26^2 + 26^3 - 85)(10^0 + 10^1 + 10^2 + 10^3 + 10^4) - 1.
\]

This assumes that the empty letter arrangement is not one of the offending 85.

7. We can first consider the case where the number displayed is an integer. Since there are 8 digits, this allows for 99999999 positive integers, 99999999 negative integers, and 0, for a total of 199999999 integers.

For non-integers, we can avoid over-counting for cases like 1.9, 1.90, 1.900, by assuming that the right-most digit is non-zero.

If there are three digits displayed, the decimal can be either ".xxx", "x.xx" or "xx.x". This results in \(10 \times 10 \times 9 \times 10 \times 9 + 9 \times 10 \times 9\) numbers.

In general, for a \(k\)-digit non-integer, there will be \(10^{k-1} \times 9 + (k - 1) \times 9 \times 10^{k-2} \times 9\) possible numbers, where the first term represents the case where the decimal leads, and the second term represents all other cases where the decimal is between digits. Since \(k\) can vary from 1 to 8, we get

\[
9 + (90 + 81) + (900 + 810 \times 2) + (9000 + 8100 \times 3) + (90000 + 81000 \times 4) + (900000 + 810000 \times 5) + (9000000 + 8100000 \times 6) + (90000000 + 81000000 \times 7).
\]

Multiplying by 2 to account for the negative numbers gives us 1440000000 non-integers. Adding this to the number of integers results in 1639999999 possible numbers.
(c) The probability of drawing a random number with two or more 6s can be calculated by subtracting the answer to (a) from the answer to (b) and dividing by the size of the sample space (100000). This results in $(100000 - 9^5 - 32805)/100000 = 8.146\%$

19. Let the sample space $S$ be the set of all permutations of the 6 employees, then $|S| = 6!$. Let $E$ be the event where the married couple are seated next to each-other. We can break $E$ down into the cases where the husband is on the left and the husband is on the right.

When the husband is on the left, there are 5 choices for his desk (he cannot occupy the right-most desk, otherwise the wife cannot be to his right), one choice for the wife’s desk, and 4! permutations of the remaining employees. When the husband is on the right, there are 5 choices for his desk, one choice for the wife’s desk, and again 4! permutations of the remaining employees. So $|E| = 2 \times 5 \times 4!$. The probability of $E$ is then $2 \times 5!/6! = 1/3$. Since the event of interest is actually the event where $E$ does not occur, the probability of the couple not being seated next to each-other is $1 - 1/3 = 2/3$.

28. Let $M$ be the set of all married people in the survey, $F$ be the set of all females, and $T$ be the set of all people between 20 and 30 years old. Then according to the survey, we have the following:

<table>
<thead>
<tr>
<th>M</th>
<th>T</th>
<th>F</th>
<th>$M \cap T$</th>
<th>$M \cap F$</th>
<th>$T \cap F$</th>
<th>$M \cap T \cap F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>675</td>
<td>682</td>
<td>684</td>
<td>195</td>
<td>467</td>
<td>318</td>
<td>165</td>
</tr>
</tbody>
</table>

By the inclusion-exclusion formula, $|M \cup T \cup F| = |M| + |T| + |F| - |M \cap T| - |M \cap F| - |T \cap F| + |M \cap T \cap F|$. Plugging in the numbers,

$675 + 682 + 684 - 195 - 467 - 318 + 165 = 1226 > 1200$,

so the figures are not consistent and could not have been the result of a real survey.

30. Since the prime factors of 1000 are 2 and 5, we are trying to calculate the number of positive integers less than 1000 which are not multiples of 2 or 5. It will be simpler to subtract the number of multiples of 2 or 5 from the total, but we will also need to consider numbers that are multiples of both 2 and 5.

Let $N_2$ be the set of numbers that are multiples of 2. There are 499 such numbers: $N_2 = \{1, 2, \ldots, 499\}$. Let $N_5$ be the set of multiples of 5: $N_5 = \{1, 2, \ldots, 199\}$, $|N_5| = 199$. Let $N_{10}$ be the intersection of $N_2$ and $N_5$. Because the least common multiple of 2 and 5 is 10, everything in $N_2 \cap N_5$ must be a multiple of 10. Therefore, $N_{10} = \{1, 2, \ldots, 99\}$.

Putting it all together, we get that $|N_2 \cup N_{10}| = 499 + 199 - 99 = 599$, so the number of numbers that are not multiples of 2 or 5 is $999 - 599 = 400$.

31. Let $A$ be the set of strings that do not start with $a$, $b$, $c$, and let $B$ be the set of strings that do not end with $c$, $d$, $e$. If $U$ is the set of all permutations of $abcde$, then the quantity of interest is $|U - A \cup B| = |U| - |A| - |B| + |A \cap B|$.

To find the size of $A$, note that the strings can only start with $d$ or $e$, and the remaining symbols can be arranged in any way. So there are 2 choices for the first symbol, 4 four the second, 3 for the third, 2 for the fourth and 1 for the fifth. $|A| = 2 \times 4 \times 3 \times 2 = 48$.

Similarly, strings in $B$ have only two choices for the last symbol (a or b), and the rest can be arranged just as in $A$, so $|B| = 4 \times 3 \times 2 \times 1 \times 2 = 48$.

Strings in $A \cap B$ must start with $d$ or $e$ and end with $a$ or $b$. So there are 2 choices for the first symbol, 2 choices for the last, and then 3! ways to arrange the remaining 3. So $|A \cap B| = 2 \times 3 \times 2 \times 1 \times 2 = 24$.

Putting it all together, we get $5! - 48 - 48 + 24 = 48$ strings.

32. Following the hint, we will let $A_i$ represent the set of numbers that don’t contain the digit $i$. If $U$ is the set of all numbers $\{1, 2, \ldots, 999999\}$, then the quantity of interest is $|U - A_1 \cup A_2 \cup A_3|$. Expanding this out, we get $|U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$. We will need to calculate the sizes of $A_1$, $A_1 \cap A_2$, and $A_1 \cap A_2 \cap A_3$, but the rest we can get by symmetry.

For $A_1$, we can handle multi-digit numbers by imagining padding with leading zeros. Thus, there are $9^6$ different combinations of digits (excluding 1). We must subtract one since 000000 isn’t positive, so $|A_1| = 9^6 - 1$. We can repeat this argument for $A_2$ and $A_3$ and arrive at the same result.

Similarly for $A_1 \cap A_2$, there are 8 choices for each digit, resulting in $8^6 - 1$ numbers. Finally, $A_1 \cap A_2 \cap A_3$ has 7 choices for each digit, resulting in $7^6 - 1$ numbers.

Putting it all together, we get $|U - A_1 \cup A_2 \cup A_3| = 9999999 - 3 \times (9^6 - 1) + 3 \times (8^6 - 1) - (7^6 - 1)$. 