No books, no calculators. Two double-sided 3x5 cards with handwritten notes allowed.

Before starting the test, please write your test number on the top-right hand corner of each page.

Name: ____________________________

Student ID:_______________________

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1. 15 pts. How many ways are there for a class of 30 students to break up into ten groups of 3?

Solution: We can first group the students into group1, group2 ... group10. This is a multinomial problem, so the answer is \( \binom{30}{3,3,3,3,3,3,3,3,3} \). However, this overcounts, because the groups aren’t distinct (group1 with students A,B,C and group2 with students D,E,F is the same as group1 with students D,E,F and group2 with students A,B,C).

We need to divide by the permutations of the groups, or 10!. So, the final answer is \( \frac{30!}{10!(3!)^{10}} \). 

2. 15 pts. We have ten students: Ann, Ann’, Bob, Bob’, Cathy, Dinar, Ephram, Frank, Giselle, Hui. Ann and Ann’ are identical twins, and so are Bob and Bob’. They’re so identical that we can’t distinguish them (for example, there’s only one way to line Ann and Ann’ up in a row, because we can’t tell the difference between Ann first and Ann’ second and vice-versa).

Ann and Ann’ are best friends and always stand next to each other.
Bob and Bob’ fight with each other and always make sure at least one person is between them.

How many ways are there to arrange these ten people in a circle?

Solution: First, make Ann/Ann’ a conjoined person (conceptually, not surgically). Now we have 9 people to arrange in a circle: 7 distinct, and two identical, and the identical people must have at least one person in-between.

Each circle has to have an AnnAnn’. Let’s consider breaking the circle before that AA’. Now we have a line starting with AA’.

In order to deal with Bob and Bob’, let’ use the complement principle.
Total number of ways to arrange everyone but Ann/Ann’ is \(\frac{8!}{2!}\) (La Jolla problem).

Total number of ways to arrange everyone but Ann/Ann’ with Bob next to Bob’: 7!

Final answer:

\[
\frac{8!}{2!} - 7!
\]
3. 15 pts. A man is dealt 2 diamond cards from an ordinary deck of 52 cards. If he is dealt 5 more cards, what is the probability that he ends up with at least 5 of his 7 cards as diamonds?

Solution: There are 50 remaining cards of which 11 are diamonds. Let’s break it into cases based on how many diamonds he ends up with:

- 5 diamonds: he had to be dealt exactly 3 diamonds out of 11, and then 2 out of the remaining 39: \( \binom{11}{3} \binom{39}{2} \).
- 6 diamonds: he had to be dealt exactly 4 diamonds out of 11, and then 1 out of the remaining 39: \( \binom{11}{4} \binom{39}{1} \).
- 7 diamonds: he had to be dealt exactly 5 diamonds out of 11: \( \binom{11}{5} \).

The probability of ending up with 5, 6, or 7 diamonds:

\[
\frac{\binom{11}{3} \binom{39}{2} + \binom{11}{4} \binom{39}{1} + \binom{11}{5}}{\binom{50}{5}}
\]
4. 20 pts. We roll five identical dice.

   (a) How many distinguishable outcomes are possible? (For example, 1,5,5,5,5 is considered to be the same as 5,5,1,5,5).

   (b) In how many outcomes are there exactly 3 different numbers showing?

*Solution:* Part a) We are assigning a count to each possible die roll of 1-6. We can view this as a bars and stars problem where we assign 5 stars (the number of dice) to 5 bars (the 6 die values). Answer is \( \binom{10}{5} \).

Part b) First, we choose the 3 different numbers from the 6 total numbers. There are \( \binom{6}{3} \) ways to do that. Then, we have a bars and stars problem again, with 5 stars and 2 bars (the 3 chosen die values). Final answer:

\[
\binom{6}{3} \binom{7}{5}
\]
5. 20 pts. How many 13-card hands are void in at least one suit (that is, have at most 3 suits)?

Solution:

Define $S_S$ to be the set of hands void in spades. Define $S_H$ to be the set of hands void in hearts. Define $S_D$ to be the set of hands void in diamonds. Define $S_C$ to be the set of hands void in clubs. Let $A_1$ be the sum of the sizes of the 4 sets. Let $A_2$ be the sum of the pairwise intersections of the 4 sets. Let $A_3$ be the sum of the three-way intersections of the 4 sets. Let $A_4$ be the sum of the four-way intersections of the 4 sets.

Our answer will be $A_1 - A_2 + A_3 - A_4$.

Each of the $S$ sets is of size $\binom{39}{13}$. $A_1$ equals $\binom{4}{1} \binom{39}{13}$.

The pairwise intersections are void in two suits. First, pick two suits ($\binom{4}{2}$), then choose 13 cards from those two suits: $\binom{26}{13}$.

The three-way intersections are void in three suits. First pick those three suits ($\binom{4}{3}$), then choose 13 cards from the remaining suit: $\binom{13}{13}$.

The fourway intersection is empty. You can’t have 13 cards with no cards in any suit.

Final answer:

\[
\binom{4}{1} \binom{39}{13} - \binom{4}{2} \binom{26}{13} + \binom{4}{3} \binom{13}{13} - 0
\]
6. 15 pts. A theater club gives 7 plays in one season. Five women in the club are cast in 3 of the plays. Prove that some play has at least 3 women in its cast.

Solution: There are fifteen total appearances by women, so the average women/play is $15/7 \approx 2.1$. By the pigeonhole principle, the max women/play $\geq 2.1$. Since the max is an integer, the max women/play $\geq 3$
7. 15 pts.  Extra Credit. Given the following 16 points on the Euclidean plan, how many triangles can be drawn whose 3 vertices are all among these points?

Solution: Let’s use the complement principle.
The total number of ways to choose 3 vertices from 16 points is \( \binom{16}{3} \).
Of those, some aren’t triangles because they’re lines. In order to be a line, they must all in a row horizontally, vertically, or diagonally.
There are 4 rows, and for each row, \( \binom{4}{3} \) ways to choose the vertices.
There are 4 columns, and for each column, \( \binom{4}{3} \) ways to choose the vertices. There are four diagonals of size 3. There are two diagonals of size 4, and for each, there are \( \binom{4}{3} \) ways to choose the vertices.
Final answer:

\[
\binom{16}{3} - \left( 10 \binom{4}{3} + 4 \right)
\]