1. Let $A, B, C$ be sets. Show that if $A \subseteq B$ then $A \cup B \subseteq B \cup C$.

**Solution.**
Let $x \in A \cup B$. Then $x \in A$ or $x \in B$ (or both).

**case 1:** $x \in A$. Since $x \in A$ and $A \subseteq B$, $x \in B$. Since $x \in B$, $x \in B \cup C$.

**case 2:** $x \in B$. Since $x \in B$, $x \in B \cup C$. 
2. Let $\mathcal{P}(X)$ be the powerset of $X = \{a, b, c\}$; that is,

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$ 

Define a binary relation $R$ on $\mathcal{P}(X)$:

$$(A, B) \in R \text{ when the number of elements of } A \text{ is less than or equal to the number of elements of } B.$$ 

For example, $(\{a\}, \{b, c\}) \in R$ (since $1 \leq 2$), but $(\{a, b\}, \emptyset) \notin R$ (since $2 > 0$).

Determine whether this relation is reflexive, symmetric, and transitive. Give a counterexample for any property that fails.

*Hint:* Recall the definitions: $R$ is

- **reflexive** if $\forall X \in \mathcal{P}(X), \ (X, X) \in R$;
- **symmetric** if $(X, Y) \in R \implies (Y, X) \in R$; and
- **transitive** if $(X, Y) \in R$ and $(Y, Z) \in R \implies (X, Z) \in R$.

**Solution.**

$R$ is reflexive, since any finite set has fewer than or the same number of elements as itself. $R$ is not symmetric; a counterexample is $(\emptyset, \{a\}) \in R$ but $(\{a\}, \emptyset) \notin R$. $R$ is transitive since if set $X$ has fewer than or the same number of elements as set $Y$, and set $Y$ has fewer than or the same number of elements as set $Z$, then $X$ has fewer than or the same number of elements as set $Z$. 