1. Four-pole Hanoi Tower: Suppose that the Tower of Hanoi problem has four poles in a row instead of three. Rings can be transferred one by one from one pole to any other pole, but at no time may a larger ring be placed on top of a smaller ring. Let \( s(n) \) be the minimum number of moves needed to transfer the entire tower of \( n \) rings from the left-most pole to the right-most pole (20 points)

a) Find \( s(1) \), \( s(2) \) and \( s(3) \)

\[
\begin{align*}
\ s(1) &= 1 \\
\ s(2) &= 3 \\
\ s(3) &= 5 \\
\end{align*}
\]

b) Explain how to do \( s(4) \)

\[
\begin{align*}
\ s(4) &= s(2) + 1 + 1 + 1 + s(2) = 9 \\
\end{align*}
\]

c) Find the recurrence representation of \( s(n) \)

\[
\ s(n) = 2s(n-2) + 3 \\
\]

d) Find \( s(6) \)

\[
\begin{align*}
\ s(6) &= 2s(4) + 3 = 2 \times 9 + 3 = 21 \\
\end{align*}
\]

2. Counting strings (20 points)

a) Make a list of all bit strings of lengths one, two, three and four that do not contain the bit pattern 000

For length one: 0, 1
For length two: 00, 01, 10, 11
For length three: 001, 010, 011, 100, 101, 110, 111
For length four: 0010, 0011, 0100, 0101, 0110, 0111, 1001, 1010, 1011, 1100, 1101, 1110, 1111

b) For each non-negative integer \( n \), let \( d(n) \) = the number of bit strings of
length n that do not contain the bit pattern 000, find d(0) to d(4)

d(0)=1

d(1)=2

d(2)=4

d(3)=7

d(4)=13

c) Find a recurrence relation for d(n)

d(n)=d(n-1)+d(n-2)+d(n-3)

d) What is d(6)?

d(5)=d(4)+d(3)+d(2)=24

d(6)=d(5)+d(4)+d(3)=44

3. Suppose for each step you take when climbing a staircase, you can move up either one or two stairs. To climb the entire staircase you can take one stair at a time or two at a time or any combination of them. Let S(n) be the number of different ways to climb a staircase with n stairs: (15 points)

a) Find S(1), S(2) and S(3)

S(1)=1

S(2)=2

S(3)=3

b) Find a recurrence relation for S(n)

S(n)=S(n-1)+S(n-2)

c) What is S(6)?

S(4)=S(3)+S(2)=5

S(5)=S(4)+S(3)=8

S(6)=S(5)+S(4)=13

4. Solve recursion by iteration (15 points)

a) \( h(k) = 2^k + h(k - 1); h(0) = 1 \) (hint: by iteration)

\( h(1) = 2^1 + h(0) = 2 + 1 \)
\[ h(2) = 2^2 + h(1) = 2^2 + 2 + 1 \]
\[ h(3) = 2^3 + h(2) = 2^3 + 2^2 + 2 + 1 \]
\[ h(k) = 2^k + 2^{k-1} + \ldots + 2 + 1 = \frac{2^{k+1} - 1}{2 - 1} = 2^{k+1} - 1 \]

b) \[ b(k) = 7b(k-1) - 10b(k-2); b(0) = 2, b(1) = 2; \text{ (hint: solving linear recursion)} \]
\[ t^2 - 7t + 10 = 0 \]
\[ (t - 2)(t - 5) = 0 \]
\[ t = 2, 5 \]
\[ b(0) = C + D = 2 \]
\[ b(1) = 2C + 5D = 2 \]
So, \[ C = \frac{8}{3}, D = -\frac{2}{3} \]
Therefore, \[ b(k) = \frac{8}{3} \cdot 2^k - \frac{2}{3} \cdot 5^k \]

c) \[ T(n) = 2T(n/2) + 5 \text{ (hint: by master method)} \]
\[ n \log_2 n = n^\alpha \]
Hence, \[ T(n) \text{ is } \Theta(n) \]

5. Show that for any real number \( x > 1 \), (10 points)
a) \[ |x^2| \leq |2x^2 + 15x + 4| \]
\[ 0 \leq x^2 + 15x + 4 \text{ because all terms are nonnegative} \]
\[ x^2 \leq 2x^2 + 15x + 4 \]
\[ |x^2| \leq |2x^2 + 15x + 4| \text{ because all terms are nonnegative} \]
b) Use the \( \Omega \) or \( O \) notation to express the results of (a)
\[ 2x^2 + 15x + 4 \text{ is } \Omega(x^2) \]

6. Use the definition of \( \Theta \)-notation to show that \( 5x^3 + 72x^2 + 30 \) is \( \Theta(x^3) \) (10 points)
For upper bound, when \( x > 1 \)
\[ |5x^3 + 72x^2 + 30| \leq 5x^3 + 72x^2 + 30 \text{ because all terms are nonnegative} \]
\[ |5x^3 + 72x^2 + 30| \leq 5x^3 + 72x^3 + 30x^3 \text{ because } 72x^2 \leq 72x^3 \text{ and } 30 \leq 30x^3 \]
\[ |5x^3 + 72x^2 + 30| \leq 107x^3 \]
\[ |5x^3 + 72x^2 + 30| \leq 107|x^3| \] since \( x^3 \) is positive when \( x > 1 \)

Let \( b=1 \) and \( B=107 \). Then \( |5x^3 + 72x^2 + 30| \leq B|x^3| \) for all real number \( x > b \)

For lower bound, when \( x > 1 \)
\[ 0 \leq 4x^3 + 72x^2 + 30 \] because all terms are nonnegative
\[ x^3 \leq 5x^3 + 72x^2 + 30 \] \( |x^3| \leq |5x^3 + 72x^2 + 30| \) because all terms are nonnegative
Let \( a=1 \) and \( A=1 \). Then \( A|x^3| \leq |5x^3 + 72x^2 + 30| \) for all real number \( x > a \)

Let \( k=\max(a,b) \). Then,
\[ A|x^3| \leq |5x^3 + 72x^2 + 30| \leq B|x^3| \] for all real number \( x > k \)
Hence, by definition of \( \Theta \)-notation, \( 5x^3 + 72x^2 + 30 \) is \( \Theta(x^3) \)

7. Determine the following \( f(n) \) is \( \Theta(?) \) (15 points)

a) \( f(n) = \frac{n(n+2)(3n+4)}{6n-2} \)
\( \frac{n(n+2)(3n+4)}{6n-2} \) is \( \Theta(n^2) \)

b) \( f(n) = 5n^3 + \frac{1}{n^2+1} + \lg n \)
\( 5n^3 + \frac{1}{n^2+1} + \lg n \) is \( \Theta(n^3) \)

c) \( f(n) = \frac{\sqrt{n(n-4)}}{n^2-2} \)
\( \frac{\sqrt{n(n-4)}}{n^2-2} \) is \( \Theta(n^{-\frac{7}{2}}) \)

8. For the following algorithm segment with input size \( n \): (15 points)

\[
\begin{align*}
&\text{for } i:=1 \text{ to } 2n \\
&\quad \text{for } j:=1 \text{ to } n \\
&\quad \quad \text{for } k:=1 \text{ to } j \\
&\quad \quad \quad a:= i*j+k \\
&\quad \quad \quad \text{next } k \\
&\quad \quad \text{next } j \\
&\quad \text{next } i \\
\end{align*}
\]

a) Compute the actual number of operations (operation within the innermost loop) that must be performed when the algorithm is executed.

For each iteration of the outermost loop, the number of iterations of the innermost loop is \( 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \). Since the outermost loop is iterated
2n times, the total number of iteration of the innermost loop is \( \frac{n(n+1)}{2} \ast 2n = n^2(n + 1) = n^3 + n^2 \). Because there are two elementary operations within each innermost loop, the total number of operations is \( 2n^3 + 2n^2 \).

b) Find an order for the algorithm segment.

\( \Theta(n^3) \)