Review – probability

- Intuition is bad, formalism is good
  - Definitions
    - Sample space: set of all possible outcomes
    - Event: subset of a sample space
  - The 4-step method – using a decision tree
    - Find the sample space
    - Define events of interest
    - Determine outcome probabilities
    - Compute event probabilities
- Probability of the union of 2 events
  - \( \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \)
- Expectation
  - \( \mathbb{E}(X) = \sum_{e \in \mathcal{S}} \Pr(e) X(e) \)
Today’s topic

- Conditional probability
- Bayes formula
- Independent events

Conditional Probability

- Pick a random person in the world
  - Event A: the person is a student in UCSD
  - Event B: the person lives in UTC area

- What’s the probability that a person is a UCSD student given that the person lives in UTC area?
Conditional Probability

• \( \text{Pr} (A \mid B) \)
  – Probability that one event (A) happens, given that some other event (B) definitely happens

  – What’s the probability that it will rain this afternoon, given that it is cloudy this morning?
  – What is the probability that two rolled dice sum to 10, given that both are odd?
  – What is the probability that I’ll get 4-of-a-kind, given that I’m initially dealt two queens?

Compute Conditional Probability

• \( \text{Pr} (A \mid B) \)
  – Probability of A given B
Example: the Tritons

- In a best-of-three tournament:
  - For the first game: the Tritons wins with probability $\frac{1}{2}$.
  - Subsequent games: probability based on the previous game
    - Won the previous game: probability of winning $\frac{2}{3}$
    - Lost the previous game: probability of winning $\frac{1}{3}$

- What's the probability of Tritons wins the series (win 2 and up games), given that they win the first game?

The Tritons: Formal approach

- Four steps:
  1. Find the sample space
  2. Define events of interest
  3. Determine outcome probabilities
  4. Compute event probabilities
The Tritons – the sample space

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Why tree diagrams work?

- \( \Pr( WW ) = \)

• Why is this correct?
  - \( \Pr( \text{win 1st game} \cap \text{win 2nd game} ) = \)\( \Pr( \text{win 1st game} ) \times \Pr( \text{win 2nd game} \mid \text{win 1st game} ) \)
  
  - \( \Pr( A_1 \cap A_2 ) = \Pr( A_1 ) \times \Pr( A_2 \mid A_1 ) \)

  - \( \Pr( A_1 \cap \ldots \cap A_n ) = \)\( \Pr( A_1 ) \times \Pr( A_2 \mid A_1 ) \times \Pr( A_3 \mid A_1 \cap A_2 ) \ldots \times \Pr( A_n \mid A_1 \cap \ldots \cap A_{n-1} ) \)
Example: Medical testing

• A deadly disease with no symptoms has infected 10% of the population. There’s only a non-perfect test for the disease:
  – If you have the disease, there’s a 10% chance that the test will say you don’t (“false negatives”)
  – If you don’t have the disease, there’s a 30% chance that the test will say you do (“false positives”)

• A random person is tested and shown positive ☑, what is the probability that this person has the disease?

Example: Medical testing

• The 4-step solution:

<table>
<thead>
<tr>
<th>Person has disease?</th>
<th>Test result</th>
<th>Outcome probability</th>
<th>Event A: Has disease?</th>
<th>Event B: Positive?</th>
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Example: Medical testing conclusion

- Good news! Only 25% chance that you have the disease!
  - Surprising?

- Probability of test being correct for any random person?

- !!! Best strategy is to ignore the test result???

Bayes’ Theorem

- Suppose a sample space $S$ is partitioned into mutually disjoint events $S_1, S_2, \ldots S_n$, and an event $A$ in $S$ has $\Pr(A) > 0$, then:

\[ Pr(S_k \mid A) = \frac{Pr(A \mid S_k) Pr(S_k)}{Pr(A \mid S_1) Pr(S_1) + Pr(A \mid S_2) Pr(S_2) + \ldots + Pr(A \mid S_n) Pr(S_n)} \]
Example: the same disease’s back!

- A deadly disease with no symptoms has infected 10% of the population. There’s only a non-perfect test for the disease:
  - If you have the disease, there’s a 10% chance that the test will say you don’t (“false negatives”)
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- A random person is tested and shown positive ☺, what is the probability that this person has the disease?

Independence

- Flip two coins simultaneously, the way one coin lands does not affect the way the other coin lands
  - A: 1\textsuperscript{st} coin comes up heads
  - B: 2\textsuperscript{nd} coin comes up heads
  - \( \Pr(A \cap B) = \Pr(A) \times \Pr(B) \)

- Cloudy & Rainy
  - C: tomorrow is cloudy (\( \Pr(C) = 1/5 \))
  - R: tomorrow is rainy (\( \Pr(R) = 1/10 \))
  - \( \Pr(R \cap C) = 1/50 ?? \)
  - Actually, not independent: every rainy day is
Independence

• Events A and B are independent iff:
  – \( \Pr(A \cap B) = \Pr(A) \times \Pr(B) \)

• Equivalently:
  – \( \Pr(A \mid B) = \Pr(A) \) (if \( \Pr(B) \neq 0 \))

• Meaning?
  – Probability of A is unaffected by the fact that B happens

Independence Intuition

• Suppose A and B are disjoint events:

• Are they independent?
Independence Intuition

• A better mental picture:

Example: Experiment with 2 coins

• Flip 2 independent fair coins. Consider the two events:
  – A: the coins match (HH or TT)
  – B: the first coin is heads

• Are A & B independent events?

• Four steps again:
  – Find the sample space
  – Define events of interests
  – Compute outcome probabilities
  – Compute event probabilities
Example: 2-coin Experiment

• The 4-step solution:

<table>
<thead>
<tr>
<th>Coin 1</th>
<th>Coin 2</th>
<th>Outcome probability</th>
<th>Event A: Coins match?</th>
<th>Event B: Coin 1 head?</th>
<th>Event A ∩ B</th>
</tr>
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Example: What about loaded coins?

• Flip 2 independent coins, not necessarily fair. Suppose each is heads with probability $p$:
  – A: the coins match (HH or TT)
  – B: the first coin is heads

• Are A & B independent events?

• Four steps again:
  – Find the sample space
  – Define events of interests
  – Compute outcome probabilities
  – Compute event probabilities
Mutual Independence

- Events A and B are independent iff:
  - \(\Pr(A \cap B) = \Pr(A) \times \Pr(B)\)

- What about more than two events?
  - How can we say that the orientations of \(n\) coins are all independent of one another?

- Events \(E_1, \ldots, E_n\) are mutually independent iff:
  - For every subset of the events, the probability of the intersection is the product of the probabilities.
  - In other words, ALL of the following equations must hold:
    - \(\Pr(E_i \cap E_j) = \Pr(E_i) \times \Pr(E_j)\) for all distinct \(i, j\)
    - \(\Pr(E_i \cap E_j \cap E_k) = \Pr(E_i) \times \Pr(E_j) \times \Pr(E_k)\) for all distinct \(i, j, k\)
    - ..... 
    - \(\Pr(E_1 \cap \ldots \cap E_n) = \Pr(E_1) \times \ldots \times \Pr(E_n)\)

Pairwise Independence

- Too many conditions for mutually dependence?

- How about pairwise independence?
  - A set of events in pairwise independence if every pair is independent

- Mutually dependence vs. pairwise dependence?
Example: Subtlety of independence

• Flip 3 fair, mutually independent coins
  – A: coin 1 and 2 match
  – B: coin 2 and 3 match
  – C: coin 3 and 1 match
• Are A, B and C mutually independent events?

Example: Subtlety of independence

• Flip 3 fair, mutually independent coins
  – A: coin 1 and 2 match
  – B: coin 2 and 3 match
  – C: coin 3 and 1 match
• Are A, B and C mutually independent events?

• Conclusion: pairwise independence is a much weaker property
Extra Example for Conditional Probability

- Suppose you have two coins:
  - One is fair: comes up heads with probability 1/2
  - The other is a trick coin: heads on both sides.
- Now you choose a coin at random, flip that coin and get heads. What is the probability that you flipped the fair coin?

A variation of the example

- If someone gives you either a fair coin or a trick coin, [flip that coin 100 times and get heads every time (!!). What is the probability that you flipped the fair coin?]