Relations

If A and B are sets, a subset, R, of A X B is a relation from A to B.

- Given relation R, if, for all a in A, there is a unique b in B such that (a, b) in R, then R is a functional relation

\[ \forall a \in A, \exists! b \in B: (a, b) \in R \]

Examples:
- \( A = B = \{-1, 1\}, R = \{(x, y): x^2 + y^2 = 1\} \)
- \( A = \{-1, 1\}, B = \{0, 1\}, R = \{(x, y): x^2 + y^2 = 1\} \)
- \( A = B = R = \{(x, y): y = x^3\} \)

Equivalence Classes

Are 1 and 101 equal?

Are there situations where we treat them as equivalent?

What other numbers are equivalent to 1 and 101 in that same situation?

Equivalence

If R is a relation from A to A (binary relation) then R is an equivalence relation if and only if R is:

- Symmetric: if \( x R y \), then \( y R x \)
- Transitive: if \( x R y \) and \( y R z \), then \( x R z \)
- Reflexive: \( x R x \)

The set of all elements equivalent to x is called the equivalence class of x.

We write \( a \equiv b \) to show that a is equivalent to b.

Example

A partition of a set induces an equivalence relation. \( x R y \) if and y are in the same block of the partition.

- Modulus induces an equivalence relation (all numbers that have the same remainder when divided by \( k \) are equivalent).
- Equality is an equivalence relation.
Showing a Binary Relation Graphically
Let $S = \{1, 2, 3, 4\}$
Let $R = \{(1, 3), (3, 1), (1, 1), (3, 3), (2, 2), (4, 4)\}$

How Many Equivalence Relations?
Given a set $S$, how many different equivalence relations are there?

Examples
Rationals
Propositional formulas

Partially Ordered Set (Poset)
Binary relation $R$ on $S$ is an order relation if it is:
- Reflexive
  - $aRa$
- Transitive
  - $aRb$ and $bRc$ implies $aRc$
- Anti-symmetric
  - $aRb$ and $bRa$ implies $a=b$

Order relations are commonly written $a \preceq b$ (precedes)
- Rather than $aRb$
- $a \prec b$ (a strictly precedes)

Combination of a set and an order relation on the set is called a partially ordered set (poset)

Example
- $(N, \preceq)$
- $S$, a set of sets, with $\subseteq$ as an order relation
Diagramming Posets

Hasse diagram of a poset (S, \( \preceq \)): graph with:
- Each element of S as a node of the graph
- Line from a to b if a immediately precedes b
- Lines always point up
- Used for project management, for instance
  - Critical chain: longest line
  - PERT (Program Evaluation and Review Technique)
  - CPM (Critical Path Method)

Example: poset \((P\{1,2,3\}, \subseteq)\)

Poset Example

Let the set be positive integers
Let the order relation be \( a \mid b \)

Poset Examples

Set is \( \mathbb{Z} \), order relation is \( a \preceq b \) if \( b = a^r \) for some positive integer \( r \)

Poset Examples

\((\emptyset, R)\) (empty set)

\((S, \emptyset)\) (empty relation)
Poset Examples

Prerequisites for CSE classes

- 8A
- 8B: requires 8A
- 12: requires 8B,
- 20: requires 12
- 21: requires 20
- 30: requires 12, 20
- 100: requires 8B, 20
- 101: requires 12, 20
- 105: requires 12, 21
- 120: requires 100, 101, 141
- 131A: requires 30, 100, 105, 131A
- 140: requires 20, 30
- 141: requires 12, 20

Ordered Subset

If \((S, \leq_s)\) is a poset, then,
- Take \(X \subseteq S\)
- Take \(\leq_X = \{a \leq b : a, b \in X\}\)

\((X, \leq_X)\) is a poset

We would normally write: \(X\) is an ordered subset of \(S\)

Example: given \((Z, |)\) take \(\{1, 2, 3, 4, 5, 6, 7, 8\} \subseteq Z\)

Product Set

Given two posets \((S, \leq_s)\), and \((T, \leq_T)\), we define the product set \((R, \leq_R)\)

- \(R = S \times T\)
- \((s, t) \leq_R (s', t')\) iff \(s \leq_s s'\) and \(t \leq_T t'\)

The product set is a poset

Example

- \((N \times N, \leq)\)

Linearly Ordered Sets

If \((S, \leq)\) is a poset, then \(a\) and \(b\) in \(S\) are comparable iff
- \(a \leq b\) or \(b \leq a\)

If \((S, \leq)\) is a poset, and each pair of elements of \(S\) are comparable, then \(S\) is a linearly ordered set (or totally ordered set)

- \(\leq\) is a total ordering of \(S\)
- \(S\) is called a chain

Examples

- \((N, \leq)\)
- \((N, |)\)
- \((\{1, 2, 32, 8, 4\}, |)\)
Maximal, Minimal, Least, Greatest

Given a poset \((S, \leq)\), and an element \(e\) in \(S\):
- \(e\) is **minimal** if there is no preceding element of \(S\) (other than \(e\)).
- \(e\) is **maximal** if \(e\) precedes no element of \(S\) (other than \(e\)).
- \(e\) is **least** if it precedes all elements of \(S\).
- \(e\) is **greatest** if it is preceded by all elements of \(S\).

Upper and lower bounds

Given a poset \((S, \leq)\) and a subset of elements \(A \subseteq S, x \in S\) is:
- an **upper bound** of \(A\) if for all \(y \in A, y \leq x\)
  - Only some subsets have upper bounds
- a **lower bound** of \(A\) if for all \(y \in A, x \leq y\)
  - Only some subsets have lower bounds
- a **least upper bound** of \(A\) if it is an upper bound of \(A\) and less than all other upper bounds of \(A\)
- a **greater lower bound** of \(A\) if it is a lower bound of \(A\) and greater than all other lower bounds of \(A\)

Example:
- \(\{3, 7, 14, 21, 40, 42\}\)

Lattice

A poset \((S, \leq)\) is a **lattice** if for pair of elements in \(S\) has a least upper bound (join) and a greatest lower bound (meet):
- Haase diagram looks like a physical lattice

Example:
- \(\{N, \leq\}\)
- \(\{S, \subseteq\}\)
- \(\{Z^+, \mid\}\)

Topological Sort of a Poset

Given a poset \((S, \leq)\), it's always possible to enumerate the elements of \(S\) in a linear ordering \(s_1, s_2, s_3, \ldots, s_{|S|}\). That linear ordering is called a **topological sort** of the poset:
- \(s_i \leq s_j \iff i < j\)

Is the topological sort unique?

Constructing a topological sort from a Hasse diagram

\[
\text{Topological-sort}(S) =
\begin{cases} S \quad & \text{if } S \text{ is empty} \\ \text{return } S \quad & \text{else} \\ e = \text{a minimal element of } S \\ \text{return Concat}(e, \text{Topological-sort}(S - e)) \end{cases}
\]
Example Topological Sort

Topological sort of prerequisites
- 8A
- 8B: requires 8A
- 12: requires 8B,
- 20: requires 12
- 21: requires 20
- 30: requires 12, 20
- 100: requires 12, 21
- 101: requires 12, 21, 100
- 105: requires 12, 21
- 120: requires 100, 101, 141
- 131A: requires 30, 100, 105
- 131B: requires 30, 100, 105, 131A
- 140: requires 20, 30
- 141: requires 140