CSE 20: Midterm Solution

July 18, 2006

Name: ______________________________

Student ID:________________________

No books and no calculators are allowed. One double-sized page of handwritten notes is allowed. If you need to make an assumption to solve a problem, state the assumption.

Problem	Score
1	/6
2	/24
3	/15
4	/15
5	/20
6	/20
Extra Credit	/15
Total	/100

1. *6 pts.* How many positive 8-bit twos-complement numbers are there?

   (a) 512
   (b) 511
   (c) 256
   (d) 255
   (e) 128
   (f) 127
   (g) 64
   (h) 63
   (i) 0
   (j) 1

   Solution: (f). The maximum value a k-bit twos-complement number can express is $2^{k-1}-1=2^7-1=127$. Thus, there are 127 positive numbers.

2. *24 pts.* Circle the true statements:

   (a) $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{Z}^+$ such that $x = y+1$.
   Solution: False. The claim is that any positive real is one more than some integer. A counter-example is $x = 1.5$. There is no y such that $1.5 = y+1$.

   (b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $x = y+1$.
   Solution: True. Every integer is one more than some other integer.

   (c) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, x = y+1$.
   Solution: False. There is no real number which is one more than every other number. If we look at an adversary argument, no matter what x we choose, the adversary could choose $y = x$, so that $x \neq y+1$.

   (d) $\forall v \in \mathbb{R}^+, \exists u \in \mathbb{R}^+$ such that $uv < v$.
   Solution: True. If $u = \frac{1}{2}$, then we are saying that half of every positive real is less than that real, which is true.

(e) $\exists u \in \mathbb{R}^+$, such that $\forall v \in \mathbb{R}^+, uv < v$.

Solution: True. If $u = \frac{1}{2}$, then we are saying that half of every positive real is less than that real.

(f) $\forall x \in \mathbb{Z}$ and $\forall y \in \mathbb{Z}, \exists z \in \mathbb{Z}$ such that $z = x - y$.

Solution: True. This is just a restatement of the fact that integers are closed under subtraction. That is, the difference of any two integers is an integer.

(g) $\forall x \in D, (P(x) \vee Q(x))$
always has the same truth value as
$(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$.

Solution: False. Let D be the people in CSE 20, $P(x)$ be the predicate "x is male", and $Q(x)$ be the predicate "x is female". The first quantified statement is true, because everyone in class is either male or female, but the second statement is false, because it is not the case that all students in class are male, nor is it the case that all students in class are female.

Thus, the two quantified statements don't always have the same truth value.

(h) $\forall x \in D, (P(x) \wedge Q(x))$
always has the same truth value as
$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$.

Solution: True. The first statement says that every x in D is in the truth set of both $P(x)$ and $Q(x)$. The second statement says that every $x \in D$ is in the truth set of $P(x)$ and that every $x \in D$ is in the truth set of $Q(x)$. The two statements thus always have the same truth value: true if every element of D is in both truth sets, false otherwise.

3. *15 pts.* Use twos-complement 7-bit arithmetic to compute 35 - 13. Show your work.

   Solution:

   $35_{10} = 0100011_2$

   $13_{10} = 0001101_2$

   $-13_{10} = 1110010_2 + 1 = 1110011_2$

   $35 - 13 = 35 + -13$

   $0100011_2 + 1110011_2 = 0010110_2$

   Check: $0010110_2 = 22$

4. *15 pts.* Convert F722 from hexadecimal to octal.

   Solution: Convert to binary, rearrange into groups of three, and then back to octal.

$$
\begin{aligned}
F722_{16} &= 1111\ 0111\ 0010\ 0010 \\
&= 1\ 111\ 011\ 100\ 100\ 010_2 \\
&= 173442_8
\end{aligned}
$$

   The solution is 173442_8.

5. *20 pts.* You are given the following truth table for S.

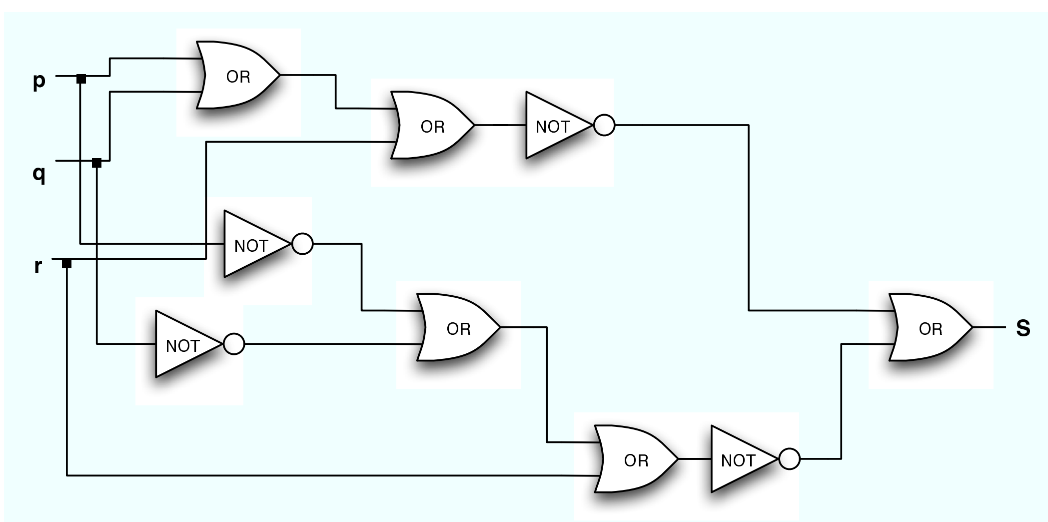
p	q	r	S
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

(a) Design a boolean function (using *only* the operators $\vee$, and $\sim$) equal to S.

Solution: We can use DNF form by creating a conjunct for each row where S is one:
$S = (\sim p \wedge \sim q \wedge \sim r) \vee (p \wedge q \wedge \sim r)$. Then, we can convert the $\wedge$ operators to $\vee$ using De'Morgan's Law: $S = \sim (p \vee q \vee r) \vee \sim (\sim p \vee \sim q \vee r)$.

(b) Draw a circuit for S using only *not* gates and *or* gates (make your *or* gates take exactly two inputs).

Solution:



6. *20 pts.* Prove that $\frac{2\sqrt(2)}{5}$ is irrational.

Solution:

Suppose $\frac{2\sqrt(2)}{5}$ were rational. Then, by the definition of rational, it could be written as the quotient of two integers p and q with $q \neq 0$.

So,

$$\begin{aligned} \frac{2\sqrt(2)}{5} &= \frac{p}{q} \\ \sqrt(2) &= \frac{5p}{2q} && \text{solving for } \sqrt(2) \end{aligned}$$

Since $5p$ is an integer (since p is an integer, and rationals are closed under multiplication), and $2q$ is a non-zero integer (since q is a non-zero integer), by the definition of rational, $\sqrt(2)$ is rational. But, as Euclid proved, $\sqrt(2)$ is irrational, a contradiction. Thus, our original assumption that $\frac{2\sqrt(2)}{5}$ is rational is incorrect.

7. *15 pts.* Extra Credit. Prove that, for any odd integer n:

$$\left\lceil \frac{n^2}{4} \right\rceil = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) + 1$$

Note: the left-hand side is using notation for the *ceiling* operator.

Solution: By the definition of odd, $n = 2k+1$ for some integer k. Note that $k = \frac{n-1}{2}$. Let's start with the left-hand side of the proposed equality:

$$\begin{aligned}
\left\lceil \frac{n^2}{4} \right\rceil &= \left\lceil \frac{(2k+1)^2}{4} \right\rceil && \text{substitution} \\
&= \left\lceil \frac{4k^2+4k+1}{4} \right\rceil && \text{multiplying out the square} \\
&= \left\lceil k^2+k+\frac{1}{4} \right\rceil && \text{algebra} \\
&= k^2+k+1 && k^2 \text{ and } k \text{ are integers} \\
&= k(k+1)+1 && \text{finding common factor} \\
&= \frac{n-1}{2}\left(\frac{n-1}{2}+1\right)+1 && \text{substituting } k = \frac{n-1}{2} \\
&= \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)+1 && \text{simplifying}
\end{aligned}$$

Thus, we've shown that the left-hand side and the right-hand side of the proposed equality are, in fact, equal.