CSE 21—Mathematics for Algorithm and System Analysis

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Day 8
Generating Functions
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Outline

What a generating function is
How to create a generating function to model a problem
Finding the desired coefficient
Partitions
Exponential generating functions

(Ordinary) Generating Functions

A different way to solve problems involving selection and arrangement problems with repetition

Can easily support special constraints not easy to do otherwise
- Find the number of ways to distribute 70 identical objects into 9 boxes with an odd number between 3 and 7 in the first box, an even number between 4 and 20 in the second box, and at most 3 in the other boxes.

Idea
- Create a generating function \( g(x) \) which models the problem of selecting \( r \) objects. The coefficient of \( x^r \) in \( g(x) \) counts the number of ways to select \( r \) objects
- \( g(x) = a_0 + a_1x + a_2x^2 + ...a_rx^r + ... + a_nx^n \)

Example
- \( g(x) = (1+x+x^2)^4 \) models the number of ways to select \( r \) objects from \( n \) objects
- \( g(x) = (1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdot\cdot\cdot + \binom{n}{r}x^r + \cdot\cdot\cdot + \binom{n}{n}x^n \)

Let’s look at the coefficients of \( x^5 \) in \( (1+x+x^2)^4 \)
- Ways to create \( x^5 \):
  - \( x^5x^x^x^x \)
  - \( x^1x^1x^1x^2 \)
  - etc.
- In general, \( x^{e_1}x^{e_2}x^{e_3}x^{e_4} \)
  - where \( 0 \leq e_1, e_2, e_3, e_4 \leq 2 \)
- Can be looked at as the number of integer solutions to:
  - \( e_1 + e_2 + e_3 + e_4 = 5 \) (where \( 0 \leq e_1, e_2, e_3, e_4 \leq 2 \))
- Which can also be looked at as the number of ways to place 5 balls in 4 boxes with at most 2 balls in each box
- Which can also be looked at as the number of ways to select 5 objects from a collection of 4 different types of objects (with two identical objects of each type).

\( g(x) = (1+x+x^2)^4 \) is a generating function for the above problems
- Actually, \( g(x) \) is a generating function for the problem with any number, \( r \), of \{balls, objects\}, not just 5
- We always write a general generating function for any \( r \)
Example

Find a generating function for \( a_r \), the number of ways to select \( r \) balls from a pile of three green, three white, three blue, and three gold balls

- Same as number of integer solutions to:
  \[- e_1 + e_2 + e_3 + e_4 = r \quad 0 \leq e_i \leq 3\]

Example

Find a generating function to model counting all selections of six objects chosen from three types of objects with:

- repetition of up to four objects of each type
- unlimited repetition

Example

Find a generating function to model:

- the number of ways to distribute \( r \) identical objects into five boxes with an even number of objects not exceeding 10 in the first two boxes and between 3 and 5 in the other boxes.

Example

Find a generating function for \( a_r \), the number of ways a roll of six distinct dice can show a sum of \( r \) if

- The first three dice are odd and the second three even
- The \( i \)th die does not show a value of \( i \)
Finding the Desired Coefficient

Reduce complicated generating function to simple function or product of simple functions

- The simple functions:
  \[
  \frac{1 - x^{m+1}}{1 - x} = 1 + x + x^2 + \cdots + x^m \\
  \frac{1}{1 - x} = 1 + x + x^2 + \cdots \\
  (1 + x)^n = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n \\
  (1 - x^{m})^n = 1 - \binom{n}{1} x^m + \binom{n}{2} x^{2m} + \cdots + (-1)^k \binom{n}{k} x^{km} + \cdots + \binom{n}{n} x^{nm} \\
  \frac{1}{(1 - x)^n} = 1 + \binom{n + 1 - 1}{1} x + \binom{2 + n - 1}{2} x^2 + \cdots + \binom{r + n - 1}{r} x^r + \cdots 
  \]

Example

Find the coefficient of \(x^{16}\) in \((x^2 + x^3 + x^4 + \cdots)^5\)

- Rewrite as product of simpler functions

  \[
  \text{Final simplification:} \\
  \text{Find coefficient}
  \]

Product of Generating Functions

If

- \(f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\)
- \(g(x) = b_0 + b_1 x + b_2 x^2 + \cdots\)

And

- \(h(x) = f(x)g(x)\)

Then

- \(h(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots + (a_0 b_r + a_1 b_{r-1} + \cdots + a_{r-1} b_1 + a_r b_0)x^r + \cdots\)

Example

Find the number of ways to collect $15 from 20 distinct people if each of the first 19 people can give $0 or $1 and the 20th person can give either $0 or $1 or $5

- Generating function: \(g(x) = \)

  \[
  \text{Simplify} \\
  \text{Find coefficient}
  \]
Example

How many ways to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls?

- Generating function
  - Simplify
  - Find coefficient

Example

How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

- Generating function
  - Simplify
  - Find coefficient

Partitions

Definition

- A partition of \( r \) identical objects divides the group into a collection (unordered) subsets of various sizes
- \((\text{Or, a group of positive integers whose sum is } r)\)

Example

- Partitions of 5

We’ll be looking at creating generating functions for partitions

Warning

- No easy way to calculate the coefficients

Generating Function for Partition

Generating function for \( a_r \)

- Want to know
  - \( e_1 \): how many 1’s
  - \( e_2 \): how many 2’s
  - …
  - \( 1e_1 + 2e_2 + \ldots + re_r = r \)

- Choose ones:
  - \((1 + x + x^2 + x^3 + \ldots + x^n + \ldots) = 1/(1-x)\)

- Choose twos
  - \((1 + x^2 + x^4 + \ldots + x^{2n} + \ldots) = 1/(1-x^2)\)

- Choose threes
  - \((1 + x^3 + x^6 + \ldots + x^{3n} + \ldots) = 1/(1-x^3)\)
  - and so on.

\( g(x) = \text{product of all those} \)

- \( 1/(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)\ldots \)
Example
Find a generating function for the number of ways to express $r$ as a sum of distinct integers

Example
Show with generating functions that every integer can be expressed as a unique sum of distinct powers of 2
- Generating function

- To show: $g(x) = 1 + x + x^2 + x^3 + \ldots$

Ferrers Diagram
Graphically displays a partition of $r$ dots
- In a set of rows of decreasing size
- Conjugate: Rotate 90 degrees counter-clockwise

Example
Show that the number of partitions of an integer $r$ as a sum of $m$ positive integers is equal to the number of partitions of $r$ as a sum of positive integers, the largest of which is $m$
Exponential Generating Functions

Used to model problems involving arrangements and distributions of distinct objects

Example problem
- find number of different arrangements of four letters, choosing from a’s, b’s, and c’s with at least two a’s.

Solution
- Four letter combinations:
  - {a, a, a, a}: \(\frac{4!}{4!0!0!}\)
  - {a, a, a, b}: \(\frac{4!}{3!1!0!}\)
  - {a, a, a, c}: \(\frac{4!}{3!0!1!}\)
  - {a, a, b, b}: \(\frac{4!}{2!2!0!}\)
  - {a, a, b, c}: \(\frac{4!}{2!1!1!}\)
  - {a, a, c, c}: \(\frac{4!}{2!0!1!}\)
- Total is sum, so we want that sum as the coefficient of \(x^4\)
- Ordinary generating function can solve number of integer solutions to:
  - \(e_1 + e_2 + e_3 = 4\) \(e_1 \geq 2\) \(e_2, e_3 \geq 0\)
- But, we want each solution to above problem to contribute:
  - \((e_1 + e_2 + e_3)! / e_1! e_2! e_3!\)

Exponential Generating Function

Function of form:
- \(g(x) = a_0 + a_1x + a_2x^2/2! + a_3x^3/3! + \ldots + a_rx^r/r! + \ldots\)

Example
- \(g(x) = (x^2/2! + x^3/3! + x^4/4! + \ldots)(1+x+x^2/2! + x^3/3! + \ldots)^2\)
- Coefficient of \(x_r\) in \(g(x)\) is equal to:
  - \(\sum_{e_1+e_2+e_3=r} \frac{1}{e_1! e_2! e_3!} \left(\frac{e_1}{e_1!} \left(\frac{e_2}{e_2!}\right) \left(\frac{e_3}{e_3!}\right)\right)\)
- Coefficient of \(x_r/r!\) in \(g(x)\) is equal to:
  - \(\sum_{e_1+e_2+e_3=r} \frac{r!}{e_1! e_2! e_3!} \left(\frac{e_1}{e_1!} \left(\frac{e_2}{e_2!}\right) \left(\frac{e_3}{e_3!}\right)\right)\)

Example

Find the exponential generating function for \(a_r\), the number of different arrangements of \(r\) objects chosen from four different type of objects with each type of object appearing at least 2 and no more than 5 times

Example

Find the number of ways to place \(r\) (distinct) people into three rooms with at least one person in each room.
Finding Desired Coefficients

Useful identities

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots \]

\[ e^{nx} = 1 + nx + \frac{n^2 x^2}{2!} + \frac{n^3 x^3}{3!} + \cdots + \frac{n^r x^r}{r!} + \cdots \]

\[ \frac{1}{2} (e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \]

\[ \frac{1}{2} (e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \]

Example

Find the number of different \( r \) arrangements of objects chosen from unlimited supplies of \( n \) types of objects

- Generating function
- Simplify
- Find coefficient

Example

Find the number of ways to place 25 people into 3 rooms with at least one person in each room

- Generating function
- Simplify
- Find coefficient

Example

Find the number of \( r \)-digit quaternary sequences (digits: 0, 1, 2, 3) with an even number of 0’s and an odd number of 1’s

- Generating function
- Simplify
- Find coefficient