Motivating Example
Create a recursive formula to specify how many ways to climb an n-stair staircase if each step covers either one or two stairsteps.

Recurrence Relation
A recursive formula counting the number of ways to do something with n objects in terms of the number of ways to do it with fewer objects.

- Example
  - \( a_n = a_{n-1} + a_{n-2} \)

General format
- \( a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ra_{n-r} \)
- \( a_n = c_1a_{n-1} + f(n) \)
- \( a_n = a_0a_{n-1} + a_1a_{n-2} + \ldots + a_{n-1}a_0 \)
- \( a_{n,m} = a_{n-1,m} + a_{n-1,m-1} \)

Must also have initial conditions.
- Example
  - \( a_0 = 0, a_1 = 3 \)

Example
A boy has n cents. Each day, he can buy one of:
- a candy bar (for 20 cents)
- a piece of gum (for 1 cent)
- an apple (for 10 cents)

How many ways are there he can buy items until the money is gone?
Example

How many $n$-digit ternary sequences without any occurrence of the subsequence 012?

Difference Equations

Sometimes easier to look at differences rather than absolutes!

- Foxes/Rabbits
  - Rabbits ($r_n$) increase each year by $a_0 r_n$ (reproduce), but foxes eat them at the rate $a_1 f_n$.
  - Foxes alone decrease each year by $a_1 f_n$, but with rabbits to eat, increase by rate $a_3 r_n f_n$.

  \[ \Delta r_n = a_0 r_n - a_1 f_n \]
  \[ \Delta f_n = a_3 r_n f_n - a_1 f_n \]

- Differential equations (continuous) = Difference equations (discrete)

First backward difference $\Delta a_n = a_n - a_{n-1}$
Second backward difference $\Delta^2 a_n = \Delta a_n - \Delta a_{n-1}$

Solving a Recurrence Relation

Bottom-up method
- Given a particular $n$ (not too big), use the recurrence relation (and initial conditions) to successively find $a_0, a_1, \ldots, a_n$.

Substitution method
- Substitute solution for $a_{n-1}$, then $a_{n-2}$, etc.
  - Example: $a_n = 100(1.08)^n$.

Guess and Verify
- Guess a solution
- Use induction to verify
  - Example: $a_n = 100(1.08)^n$.

Solving a Recurrence Relation

Given a recurrence relation of form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r}$

Rewrite as powers:

$\alpha^n = c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \ldots + c_r \alpha^{n-r}$

Simplify

$\alpha^n - c_1 \alpha^{n-1} - c_2 \alpha^{n-2} - \ldots - c_r = 0$

characteristic equation

Solve for $r$ roots, $\alpha_1, \ldots, \alpha_r$

Generate form of final solution ($a_n = \alpha^n$ is a solution to rec. relation)

$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \ldots + A_r \alpha_r^n$

Use initial conditions to create a set of simultaneous equations

$a_0 = A_1 \alpha_1^0 + A_2 \alpha_2^0 + \ldots + A_r \alpha_r^0$

Use...

Solve for $A_1, \ldots, A_r$. 
Example

Give closed form for:
- \(a_n = 1.08a_{n-1}\)
- \(a_0 = 500\)

Rewrite as powers
- \(a^n = 1.08a^{n-1}\)

Simplify
- \(\alpha - 1.08 = 0\)

Solve for \(\alpha_1\)
- \(\alpha_1 = 1.08\)

Generate form of final equation
- \(a_n = A_1\alpha_1^n\)

Use initial conditions
- \(a_0 = 500 = A_1\)

Final equation
- \(a^n = 500(1.08)^n\)

Example

Find a closed form for the number of ways to make a pile of \(n\) chips using red, white and blue chips and so that no two red chips are together
- \(a_n = \) 
- \(a_0 = \) 
- \(a_1 = \)

Example

Find a closed form for the Fibonacci sequence
- \(a_n = a_{n-1} + a_{n-2}\)
- \(a_0 = a_1 = 1\)

Adjustment for Multiple Roots

Given a recurrence relation of form
- \(a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_r a_{n-r}\)
- \(\ldots\)

Solve for \(r\) roots, \(\alpha_1, \ldots, \alpha_r\)
If we have a root \(\alpha_i\) of multiplicity 2, general form of equation is
- \(a_n = A_1\alpha_1^n + A_2\alpha_2^n + \ldots + A_i\alpha_i^n + A_{i+1}n\alpha_i^n + \ldots + A_r\alpha_r^n\)
Example with Multiple Roots

Given:
- \(a_n = -2a_{n-2} - a_{n-4}\)
- \(a_0 = 0\)
- \(a_1 = 1\)
- \(a_2 = 2\)
- \(a_3 = 3\)

Find a closed form for \(a_n\).

Divide & Conquer Recurrence Relations

If there are \(n\) (a power of two) players in a single-elimination tournament, how many rounds need to be played?
- \(a_n = a_{n/2} + 1\)

Solving Recurrence Relations

Given:
- \(a_n = cn^{k} + f(n)\)

Three general cases (in the long run):
- \(f(n)\) grows faster than \(n^{\log_k c}\)
  - \(a_n\) grows proportional to \(f(n)\)
- \(f(n)\) grows slower than \(n^{\log_k c}\)
  - \(a_n\) grows proportional to \(n^{\log_k c}\)
- \(f(n)\) grows proportional to \(n^{\log_k c}\)
  - \(a_n\) grows proportional to \(\log n \cdot n^{\log_k c}\)

Example

Merge-sort algorithm for list of \(n\) (power of 2) items:
- break list into two equal parts: L1, L2
- recursively sort L1 and L2
- Merge L1 and L2 together into L (takes 2n-1 operations)

Recurrence relation
- \(a_n = \) 

How fast does it grow?
Solving a Recurrence Relation

Given a recurrence relation of form
\[ a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ra_{n-r} + f(n) \]

Remove \( f(n) \) and find general solution:
\[ a_n = A_1^n + A_2^n + \ldots + A_r^n + p(n) \]

Use table to find form of particular solution for inhomogenous eqn.
\[ a_n = p(n) = c_1p(n-1) + \ldots + c_rp(n-r) + f(n) \]

Solve for coefficient(s) in \( p(n) \)

Modify general solution by adding \( p(n) \)
\[ a_n = A_1^n + A_2^n + \ldots + A_r^n + p(n) \]

Solve for coefficient(s)

Special case: \( c=1, r = 1 \)
\[ a_n = \Sigma f(n) \]

Example

Find closed form for
\[ a_n = 2a_{n-1} + 1 \]
\[ a_1 = 1 \]

General form for homogenous part
\[ a_n = A2^n \]

Form of particular solution
\[ p(n) = B \]
\[ a^n = p(n) = B = 2a^{n-1} + 1 = 2B + 1 \]
\[ B = -1 \]

General form with particular solution added:
\[ a_n = A2^n - 1 \]
\[ A = 1 \]

Example

Find closed form for
\[ a_n = 3a_{n-1} - 4n + 3\cdot2^n \]

General form of homogenous part
\[ a^n = A3^n \]

Form of first inhomogenous term
\[ p(n) = B_1n + B_0 \]
\[ p(n) = B_1n + B_0 = a^n = 3a_{n-1} - 4n = 3(B_1(n-1) + B_0) - 4n \]

Form of second inhomogenous term
\[ q(n) = B\cdot2^n \]
\[ q(n) = B\cdot2^n = a^n = 3a_{n-1} + 3\cdot2^n = 3B\cdot2^{n-1} + 3\cdot2^n \]

Final general solution
\[ a_n = A3^n + p(n) + q(n) \]

Final solution
Example

**Given:**
- the average of 2 successive years' production $1/2(a_n + a_{n-1})$ is $2n + 5$
- $a_0 = 3$

**Find** $a_n$