Test 1 Solutions

The following are solutions for one of the versions of the exam. Since the other versions are very similar, separate solutions will not be provided.

Problem 1 [12 points]
Indicate whether the following statements are TRUE (T) or FALSE (F) by circling your answers. Justify your answers briefly.

a) It is possible to define a DFA with no final states.
   TRUE; The set of final states of a DFA can be any subset of the set of states. In particular, it can be ∅.

b) For any languages $A$ and $B$ such that $A \subseteq B$, if $B$ is regular then $A$ is regular.
   FALSE; Let $A = \{ 0^n1^n | n \in \mathbb{N} \}$ and $B = \{0,1\}^*$. Then $B$ is regular and $A \subseteq B$, but $A$ is nonregular.

c) If the pumping lemma for regular languages holds for $L$ then $L$ is regular.
   FALSE; There exist nonregular languages (e.g., $\{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$) for which the conditions of the pumping lemma hold.

d) All finite languages are regular.
   TRUE; For any finite language $L = \{s_1, \ldots, s_k\}$, where $k \geq 1$, the regular expression $s_1 \cup \cdots \cup s_k$ describes $L$. The regular expression $\emptyset$ describes the language $\emptyset$.

e) An NFA without ε-transitions is not as powerful as an NFA with ε-transitions (i.e., there exists a language that can be recognized by some NFA with ε-transitions but cannot be recognized by any NFA without ε-transitions).
   FALSE; For any NFA (with or without ε-transitions) there exists an equivalent DFA, which is an NFA with no ε-transitions.

f) For every NFA there exists a PDA that recognizes the same language.
   TRUE; The NFA can easily be converted into a PDA that doesn’t use its stack. This PDA will recognize the same language.
Problem 2 [12 points]
Let \( L = \{0^{20}1^{20}\} \).

a) Adriana says she has a DFA with 15 states that recognizes \( L \). Explain why she is incorrect.
   Since \( |0^{20}1^{20}| > 15 \), any path in a DFA with 15 states labelled by the string \( 0^{20}1^{20} \) must revisit
   some state. If the DFA does not accept this string, then the language recognized by the DFA
   is not \( L \). If the DFA accepts the string, then there is a cycle on a path between the start
   state and a final state. Therefore, the DFA accepts infinitely many strings, and the language
   recognized by it is not \( L \).

b) What is the minimum number of states in a DFA that recognizes \( L \)? \( 42 \)
   (41 states are required to accept a string of length 40. Additionally, a “black hole” state is
   needed to reject unwanted strings.)

c) Given a DFA, how would you determine whether the language it recognizes is infinite?
   If the DFA contains a cycle that is reachable from the start state and such that a final state
   is reachable from some state in the cycle, then the language recognized by the DFA is infinite.
   Otherwise, it is finite.

Problem 3 [14 points]
Let \( \Sigma = \{0,1\} \) and let \( L = \{ w \in \Sigma^* \mid w \text{ contains an even number of } 0 \text{s, or exactly two } 1 \text{s} \} \).
In the box below, write a regular expression that describes the language \( L \).

\[
1^* (1^*01^*01^*)^* \cup 0^*10^*10^*
\]

Problem 4 [14 points]
Let \( L = \{0^m1^n \mid m+n \text{ is even} \} \).
Draw the state diagram of a DFA with at most 6 states that recognizes \( L \).
Problem 5 [10 points]
Consider the operation $\&$ defined as follows:

$$A \& B = \{ w \mid \text{there exist } x \in A, y \in B \text{ such that } w = xy \text{ and } |x| = |y| \}$$

Is the class of regular languages closed under $\&$? If so, prove it. If not, disprove it (i.e., give an example of two regular languages $A$ and $B$ such that $A \& B$ is not regular).

The class of regular languages is NOT closed under $\&$. Consider $A = \{0\}^*$ and $B = \{1\}^*$. Both of these languages are regular, but $A \& B = \{0^n1^n \mid n \in \mathbb{N} \}$ is not regular.

Problem 6 [15 points]
Let $L = \{ w \in \{0,1\}^* \mid w \text{ is an even-length palindrome} \}$.
Prove that $L$ is nonregular.

Proof by contradiction. Assume that $L$ is regular. Let $p$ be given by the pumping lemma. Let $s = 0^p110^p$. Then $s \in L$ (s is a palindrome and the length of $s$ is even) and $|s| \geq p$. By the pumping lemma, $s$ can be split into $x, y, z$ such that 1) $|y| > 0$, 2) $|xy| \leq p$, and 3) for all $i \geq 0$: $xy^iz \in L$. By condition 2), $x = 0^j$ for some $j \geq 0$ and $y = 0^k$ for some $k \geq 0$. Therefore, $z = 0^{p-j-k}110^p$. By condition 1), $k \geq 1$. If $i = 2$, then $xy^2z = xyyz = 0^j0^k0^k0^{p-j-k}110^p = 0^{p+k}110^p$. Since $k \geq 1$, $0^{p+k}110^p$ is not a palindrome. Therefore, $0^{p+k}110^p \notin L$. This contradicts condition 3).

Problem 7 [14 points]
Let $L = \{ 0^m1^n \mid m > n \}$.
Prove that $L$ is context-free by drawing the state diagram of a PDA with at most 3 states that recognizes $L$. 

![State Diagram]

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Problem 8 [14 points]
Let $\Sigma = \{0\}$ and consider the following operation defined on languages over $\Sigma$.
\[
\text{DOUBLE}(L) = \{ x \in \Sigma^{*} \mid \text{there exists } y \in L \text{ such that } |x| = 2|x| \}\]

Prove that if $L$ is regular then DOUBLE$(L)$ is also regular. Describe the idea of your construction informally and provide a formal definition. You are not required to prove that your construction is correct, but do indicate what you would have to prove to establish this.

**Given:** $L$ is regular, i.e., there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L$.

**Want:** An NFA $N$ that recognizes DOUBLE$(L)$.

**Construction:**

**Idea:** We build an NFA $N$ by replacing each transition in $M$ from a state $q_i$ to a state $q_j$ with two transitions: one from $q_i$ to a new state $q'_i$ and another from $q'_i$ to $q_j$. All paths from the start state $q_0$ to a final state in $M$ become twice as long. Therefore, $N$ will accept all strings that have twice the length of some string accepted by $M$. Notice that since $M$ is a DFA and the alphabet $\Sigma$ is unary, each state in $M$ has exactly one transition going to another state. Thus the number of additional states required is the number of states in $M$.

**Formal definition:** Let $N = (Q', \Sigma', \delta', q'_0, F')$, where each component is defined as follows.

- $Q' = \{q_0, \ldots, q_{k-1}, q'_1, \ldots, q'_k\}$, where $Q = \{q_0, \ldots, q_{k-1}\}$ and $q'_1, \ldots, q'_k \not\in Q$.
- $\Sigma' = \Sigma = \{0\}$
- $\delta' : Q' \times (\Sigma' \cup \{\varepsilon\}) \to \mathcal{P}(Q')$ is defined as follows:
  \[
  \begin{align*}
  \delta'(q_i, 0) &= \{q'_{i+1}\} & \text{for } 0 \leq i \leq k - 1 \\
  \delta'(q'_i, 0) &= \{\delta(q_i, 0)\} & \text{for } 0 \leq i \leq k - 1 \\
  \delta'(q, \varepsilon) &= \emptyset & \text{for all } q \in Q'
  \end{align*}
  \]

- $q'_0 = q_0$
- $F' = F$

**Correctness:** We must show that if $w \in \text{DOUBLE}(L)$ then $N$ accepts $w$, and if $w \not\in \text{DOUBLE}(L)$ then $N$ rejects $w$. 