Lecture 4 Overview

- Regular operations on languages
- Closure properties of the class of regular languages
- Regular expressions
- Equivalence of regular expressions and finite automata

Regular Operations on Languages

Def: Let $L$, $L_1$ and $L_2$ be languages over alphabets $\Sigma$, $\Sigma_1$, and $\Sigma_2$, respectively. The regular operations union, concatenation, and star are defined as follows.

Union: $L_1 \cup L_2 = \{w \in \Sigma_1^* \cup \Sigma_2^* \mid w \in L_1 \text{ or } w \in L_2\}$

Concatenation: $L_1 \cdot L_2 = L_1L_2 = \{w \in (\Sigma_1 \cup \Sigma_2)^* \mid w = w_1w_2 \text{ for some } w_1 \in L_1, w_2 \in L_2\}$

Star: $L^* = \{w \in \Sigma^* \mid w = w_1 \ldots w_n \text{ for some } n \geq 0, w_1, \ldots, w_n \in L\}$
Closure Properties of the Class of Regular Languages

Theorem: If $L$, $L_1$, and $L_2$ are regular languages over an alphabet $\Sigma$, then the following languages are also regular.

1) $\overline{L} = \{w \in \Sigma^* | w \notin L\}$
2) $L_1 \cup L_2 = \{w \in \Sigma^* | w \in L_1 \text{ or } w \in L_2\}$
3) $L_1 \cap L_2 = \{w \in \Sigma^* | w \in L_1 \text{ and } w \in L_2\}$
4) $L_1 \cdot L_2 = \{w \in \Sigma^* | w = w_1w_2 \text{ for } w_1 \in L_1, w_2 \in L_2\}$
5) $L^* = \{w \in \Sigma^* | w = w_1 \cdots w_n \text{ for } n \geq 0, w_1, \ldots, w_n \in L\}$

Before proving this, let’s consider some examples.

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Closure Properties – Examples

Let $\Sigma = \{0, 1\}$.

Ex. 1)
$L = \{w \in \{0, 1\}^* | w \text{ does not contain substring 1010}\}$
Is $L$ regular?
Closure Properties – Examples

Ex. 2) Prove that the following language is regular. 
\[ L = \{ w \mid w \text{ starts with 110 and contains substring 1010} \} \]

Closure Properties – Proof of Theorem

Recall the theorem we proved yesterday:

**Theorem:** A language \( L \) is regular if and only if there exists an NFA that recognizes \( L \).

We will use this result.
Closure Properties – Proof of Theorem

Proof of Theorem – part 1):

Given: $L$ is regular, i.e., there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L$.

Want: $\overline{L} = \{w \in \Sigma^* \mid w \notin L\}$ is regular, i.e., there exists an NFA $N$ that recognizes $\overline{L}$. We will build $N$. (We could build a DFA instead, but NFAs are often simpler.)

Construction: $N = (Q', \Sigma, \delta', q'_0, F')$, where

$Q' = \hdots$
$\delta' = \hdots$
$q'_0 = \hdots$
$F' = \hdots$

Closure Properties – Proof of Theorem

Proof of Theorem – part 2):

Given: $L_1$ and $L_2$ are regular, i.e., there exist DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ that recognize $L_1$ and $L_2$, respectively.

Want: $L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \text{ or } w \in L_2\}$ is regular, i.e., there exists an NFA $N$ that recognizes $L_1 \cup L_2$. We will build $N$.

Construction: $N = (Q, \Sigma, \delta, q_0, F)$, where

$Q = \hdots$
$\delta = \hdots$
$q_0 = \hdots$
$F = \hdots$
Closure Properties – Proof of Theorem

Nondeterminism makes the construction easy!
Construction: \( N = (Q, \Sigma, \delta, q_0, F) \), where

Proof of Theorem – part 3):

Given: \( L_1 \) and \( L_2 \) are regular, i.e., there exist DFAs \( M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \) and \( M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \) that recognize \( L_1 \) and \( L_2 \), respectively.
Want: \( L_1 \cap L_2 = \{ w \in \Sigma^* \mid w \in L_1 \text{ and } w \in L_2 \} \) is regular, i.e., there exists an NFA \( N \) that recognizes \( L_1 \cap L_2 \).

We could build \( N \), but there is an easier way to prove that \( L_1 \cap L_2 \) is regular.
Closure Properties – Proof of Theorem

Given: $L_1$ and $L_2$ are regular.
Want: $L_1 \cap L_2$ is regular.

Closure Properties – Proof of Theorem

Proof of Theorem – part 4):

Given: $L_1$ and $L_2$ are regular, i.e., there exist DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ that recognize $L_1$ and $L_2$, respectively.

Want:
$L_1 \cdot L_2 = \{w \in \Sigma^* | w = w_1w_2 \text{ for some } w_1 \in L_1, w_2 \in L_2\}$
is regular, i.e., there exists an NFA $N$ that recognizes $L_1 \cdot L_2$. We will build $N$. 
Closure Properties – Proof of Theorem

Again, nondeterminism makes the construction easy!
Construction: \( N = (Q, \Sigma, \delta, q_0, F) \), where

Closure Properties – Examples

Ex. 3) Consider the following languages.

\( L_1 = \{ w \mid w \text{ starts with 10} \} \)
\( L_2 = \{ w \mid w \text{ starts with 111} \} \)

By part 2) of the theorem,
\( L_1L_2 = \)

is regular.
Closure Properties – Examples

Ex. 4) Let \( L = \{00, 01, 10, 11\} \). Then

\[
L^* = \{ w_1 \cdots w_n | n \geq 0 \text{ and } w_1, \ldots, w_n \in L \} = \]

By part 5) of the theorem, \( L^* \) is regular.

Closure Properties – Proof of Theorem

Proof of Theorem – part 5): Given: \( L \) is regular, i.e., there exists a DFA \( M = (Q, \Sigma, \delta, q_0, F) \) that recognizes \( L \).

Want:
\( L^* = \{ w \in \Sigma^* | w = w_1 \cdots w_n \text{ for } n \geq 0, w_1, \ldots, w_n \in L \} \) is regular, i.e., there exists an NFA \( N \) that recognizes \( L^* \). We will build \( N \).

Construction: \( N = (Q', \Sigma, \delta', q'_0, F') \), where
\[
Q' = \quad \delta' = \quad q'_0 = \quad F' = \]
Closure Properties – Proof of Theorem

Construction: $N = (Q', \Sigma, \delta', q'_0, F')$, where

Regular Expressions