Sample Quiz 7

1. Suppose the Tower of Hanoi has 5 poles instead of three. A disk can be transferred from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk.

Let $S_n$ be the minimum number of moves to transfer the entire tower of $n$ disks from the leftmost to the rightmost pole.

Find $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$

Let the notation $m[n,s,d]$ indicate the minimum number of moves to transfer $n$ disks from pole $s$ to pole $d$.

$$S_1 = m[1,A,E] = 1$$
$$S_2 = m[1,A,D] + S_1 + m[1,D,E] = 1+1+1 = 3$$
$$S_3 = m[1,A,C] + S_2 + m[1,C,E] = 1+3+1 = 5$$
$$S_4 = m[1,A,B] + S_3 + m[1,B,E] = 1+5+1 = 7$$
$$S_5 = m[2,A,B] + S_3 + m[2,B,E] = 3+5+3 = 11$$
$$S_6 = m[3,A,B] + S_3 + m[3,B,E] = 5+5+5 = 15$$
$$S_7 = m[4,A,B] + S_3 + m[4,B,E] = 7+5+7 = 19$$

2. Prove the following for the Fibonacci Sequence $F_0, F_1, F_2, ...$

$$F_{k+1}^2 + F_{k-1}^2 = 2 F_k F_{k-1} + F_k^2, \quad k \geq 1$$

We know that $F_k = F_{k+1} - F_{k-1}$.

Solving for $F_k$ we get $F_k = F_{k+1} - F_{k-1}$.

Squaring both sides, we get $F_k^2 = F_{k+1}^2 - 2 F_{k+1} F_{k-1} + F_{k-1}^2$.

Now the proof is just a substitution away:

$$F_{k+1}^2 + F_{k-1}^2 = 2 F_{k+1} F_{k-1} + F_k^2, \quad k \geq 1$$

$$F_{k+1}^2 + F_{k-1}^2 = 2 F_{k+1} F_{k-1} + F_{k+1}^2 - 2 F_{k+1} F_{k-1} + F_{k-1}^2 + F_{k-1}^2 \text{ by substitution}$$

$$F_{k+1}^2 + F_{k-1}^2 = F_{k+1}^2 + F_{k-1}^2$$
3. Use iteration to guess an explicit formula for the following recursively defined sequence. Use the formulas from Epp (4.2) to simplify your answers whenever possible.

\[ b_k = 3 \cdot b_{k-1} + 4, \text{ for } k \geq 1 \]
\[ b_0 = 2 \]

\[ b_k \] (by defn.) = \[ 3 \cdot b_{k-1} + 4 \]

(by subst) = \[ 3 \cdot (3 \cdot b_{k-2} + 4) + 4 \]

(by simp) = \[ (3^2) \cdot b_{k-2} + (3)4 + 4 \]

(by subst) = \[ (3^2) \cdot (3 \cdot b_{k-3} + 4) + (3)4 + 4 \]

(by simp) = \[ (3^2)^2 \cdot b_{k-4} + (3^2)4 + (3)4 + 4 \]

(by subst) = \[ (3^3) \cdot (3 \cdot b_{k-4} + 4) + (3^2)4 + (3)4 + 4 \]

(by simp) = \[ (3^4) \cdot b_{k-4} + (3^3)4 + (3^2)4 + (3)4 + 4 \]

\[ \vdots \]

\[ b_k = (3^k) \cdot b_{k-k} + (3^{k-1})4 + (3^{k-2})4 + \ldots + (3^2)4 + (3)4 + 4 \]

\[ b_k = (3^k)b_0 + 4 \sum_{i=0}^{k-1} 3^i \]

\[ b_k = (3^k)(2) + 4 \sum_{i=0}^{k-1} \frac{3^i}{3-1} \]

\[ b_k = 2 \cdot (3^k) + 2 \cdot (3^k - 1) \]

\[ b_k = 2 \cdot (3^k) + 2 \cdot (3^k) - 2 \]

\[ b_k = 4 \cdot (3^k) - 2 \]