1 RW-expanders are LS-expanders

This homework is a guided walkthrough to completing Theorem 7.7 from the notes by proving that a random-walk expander is a local-spectral expander:

**Theorem 1.1.** Let \((X, \Pi)\) be a weighted \(d\)-dimensional pure simplicial complex. Then:

1. If \(X\) is a \(\gamma\)-local spectral expander then it is a \(\gamma\)-random walk expander.
2. If \(X\) is a \(\gamma\)-random walk expander then it is a \(2(d-1)\gamma\)-local spectral expander.

We break the proof into a number of steps. For any \(0 \leq i \leq d-2\) and \(\tau \in X(i)\), we want to show that \(A_\tau\) (the weighted adjacency matrix underlying \(X_\tau\)) is a \(2(d-1)\gamma\)-spectral expander. First, check the result holds for \(i = 0\):

1. Prove the result for \(i = 0\), \(\tau = \emptyset\).

Now consider \(1 \leq i \leq d-2\). By the variational characterization of eigenvalues, it is enough to bound the Rayleigh quotient. Our goal is now to prove that for any \(\tau \in X(i)\) and \(f \in C_1(X_\tau)\) such that \(f \perp \vec{1}\):

\[
\frac{|\langle A_\tau f, f \rangle|}{\langle f, f \rangle} \leq 2(i+1)\gamma \leq 2(d-1)\gamma
\]

We’ll prove this by analyzing a lifted version of \(f\) called \(\tilde{f}\in C_{i+1}(X)\) over the full complex:

\[
\tilde{f}(\sigma) = \begin{cases} f(\sigma \setminus \tau) & \text{if } \tau \subset \sigma \\ 0 & \text{else} \end{cases}
\]

Similar to our proof of 1 in class, it will be useful to understand the connection between the global function \(\tilde{f}\) and the local function \(f\). Prove the following equivalences:

2. \(\langle \tilde{f}, \tilde{f} \rangle = \pi_i(\tau)(i+1)\langle f, f \rangle\)
3. \(\langle M_{i+1}^+ \tilde{f}, \tilde{f} \rangle = \pi_i(\tau)\langle A_\tau f, f \rangle\)

Similar to the forward direction, we also need a statement about the quadratic form of the lower walk:

\[
|\langle U_i D_{i+1} \tilde{f}, \tilde{f} \rangle| \leq \gamma \langle \tilde{f}, \tilde{f} \rangle \quad (1)
\]

Since this is a bit trickier to show, we’ll prove it in the next section. Assuming Equation (1), use parts 2 and 3 to complete the result:

4. Show the Rayleigh quotient is bounded by \(2(i+1)\gamma\):

\[
\forall 1 \leq i \leq d-2, \tau \in X(i) : \frac{|\langle A_\tau f, f \rangle|}{\langle f, f \rangle} \leq 2(i+1)\gamma
\]
1.1 Analyzing the lower walk

We will now prove Equation (1), using the same definition of $\tilde{f}$ as in the previous problem. First, show that we don’t have to worry about $\langle U_i D_{i+1} \tilde{f}, \tilde{f} \rangle$ being negative:

5. Prove that $\langle U_i D_{i+1} \tilde{f}, \tilde{f} \rangle \geq 0$.

Then it is enough show:

$$\langle U_i D_{i+1} \tilde{f}, \tilde{f} \rangle \leq \gamma \langle \tilde{f}, \tilde{f} \rangle$$

Recall from class that we may write:

$$\langle U_i D_{i+1} \tilde{f}, \tilde{f} \rangle = E_{\sigma \sim \pi_i} \left[ E_{v, w \sim \pi_{i+1}} [\tilde{f}(\sigma \cup v) \tilde{f}(\sigma \cup w)] \right]$$  (2)

Prove that we can ignore the case $\sigma = \tau$, that is:

6. Prove that $E_{v, w \sim \pi_{i+1}} [\tilde{f}(\tau \cup v) \tilde{f}(\tau \cup w)] = 0$

Next, notice that if $\sigma \neq \tau$, then the only time $\tilde{f}(\sigma \cup v) \tilde{f}(\sigma \cup w)$ can be non-zero is if $\sigma \cup v = \sigma \cup w = \sigma \cup \tau$. Use this fact to simplify and bound Equation (2) by:

7. $\langle U_i D_{i+1} \tilde{f}, \tilde{f} \rangle \leq \sum_{\rho \in X(i+1)} \pi_{i+1}(\rho) \tilde{f}(\rho)^2 U_i D_{i+1} 1_{\rho}(\rho)$

(Hint: write out Equation (2) as a sum, remove cross terms by observation, and switch the summation to be over $X(i+1)$. The last step is to note that $U_i D_{i+1} 1_{\rho}(\rho)$ is small:

8. Prove that $\forall \rho \in X(i) : U_i D_{i+1} 1_{\rho}(\rho) \leq \gamma$ and conclude the result.