Problem 1: Flag Complex

The goal of this problem is to prove that the Flag Complex of $\mathbb{F}_q^d$ is a one-sided $O\left(\sqrt{\frac{1}{d}}\right)$-spectral expander so long as $d \ll \sqrt{q}$. Recall from the notes:

**Definition** (Flag Complex on $\mathbb{F}_q^d$). Let $q$ be a prime power. A **complete flag** of $\mathbb{F}_q^d$ is a strict containment sequence of $d-1$ subspaces $\{0\} \subset V_1 \subset \ldots \subset V_{d-1} \subset \mathbb{F}_q^d$. Let $Gr(d,q)$ denote the set of subspaces of $\mathbb{F}_q^d$ of any dimension. The Flag Complex on $\mathbb{F}_q^d$ is the $(d-1)$-dimensional simplicial complex on vertex set $Gr(d,q)$ whose top level faces are given by the complete flags.

**Part (a): Bipartite Spectral Expanders**

The Flag Complex can be viewed as a multipartite complex. It will therefore be useful to understand some properties of bipartite spectral expanders to start. Given a bipartite graph $G = (L, R, E)$, let

$$M = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

be the (square) adjacency matrix indexed by $L \cup R$. In the next part, we will see that it is easier to analyze the spectrum of the two-step random walks $A^T A$ and $AA^T$ rather than $M$ directly. In this part, you will show the two are closely related. In particular, show that:

1. If $\lambda$ is an eigenvalue of $M$, then $\lambda^2$ is an eigenvalue of $A^T A$ and $AA^T$.
2. If $\lambda \neq 0$ is an eigenvalue of $A^T A$ or $AA^T$, then $\pm \sqrt{\lambda}$ are eigenvalues of $M$.

**Part (b): Flag Complex in Two Dimensions**

Next, we analyze the Flag Complex of $\mathbb{F}_q^3$, which is a bipartite graph $G = (L, R, E)$ where $L$ is given by the set of lines in $\mathbb{F}_q^3$ (1-dimensional subspaces, generated by a nonzero element) and $R$ is the set of planes (2-dimensional subspaces, generated by a pair of nonzero elements which are not a multiple of each other). We will analyze $G$ through the two-step random walk from lines to planes to lines.

1. Prove that there are $q^2 + q + 1$ distinct lines and $q^2 + q + 1$ distinct planes in $\mathbb{F}_q^3$.
2. Prove that every plane contains $q + 1$ distinct lines, and every line is adjacent to $q + 1$ distinct planes.
3. Using these facts, prove that the two-step walk $A^T A$ has the following form:

$$A^T A = \frac{q}{(q+1)^2} I + \frac{1}{(q+1)^2} J$$

where $I$ is the identity matrix and $J$ is the all ones matrix.
4. Combine this with Part (a) to prove that $G$ is a one-sided $\frac{1}{\sqrt{q}}$-spectral expander.
Part (c): Flag Complex in \( d \) dimensions

We are ready to prove the main result. By Oppenheim’s theorem, it is enough to prove that all links are connected and that \((d-3)\)-links are good one-sided expanders. Let’s start by proving every link of dimension less than \( d - 3 \) is connected (OPTIONAL\(^1\)). We break the proof into two parts:

1. Prove that \( X_\emptyset \) is connected.
2. Given \( j > 2 \) and \( \sigma \in X(d-j-1) \), let \( i_1 \leq \ldots \leq i_j \) denote the dimensions missing from \( \sigma \).
   (a) Prove that if there exists a gap, i.e. \( \ell \) such that \( i_\ell + 1 < i_{\ell+1} \), \( X_\sigma \) is connected.
   (b) Otherwise, prove the link is connected by reducing to 1.

We now turn our attention to \((d-3)\)-links (REQUIRED). Every \( \sigma \in X(d-3) \) has the following structure:

\[
\sigma = \{ V_1 \subset V_2 \subset \ldots \subset V_{i-1} \subset V_{i+1} \subset \ldots \subset V_{j-1} \subset V_{j+1} \subset \ldots \subset \mathbb{F}_q^d \},
\]

i.e. a complete flag missing two arbitrary subspaces of dimensions \( i \) and \( j \). There are two cases of interest:

1. Prove that when \( j = i + 1 \), \( X_\sigma \) is isomorphic to the 2-dimensional Flag Complex (hint: first consider \( i = 1, j = 2 \), then try to reduce general \( i, j \) to this case).
2. Prove that when \( j > i + 1 \), \( X_\sigma \) is a complete bipartite graph with \( q + 1 \) vertices on each side.

Finally, assuming \( d \ll \sqrt{q} \), combine these facts with Part (b) to prove that the flag complex on \( \mathbb{F}_q^d \) is a one-sided \( O\left(\sqrt{\frac{1}{q}}\right) \)-spectral expander.

**Problem 2: Oppenheim’s Theorem**

Prove Oppenheim’s Trickling-down Theorem for two-sided local spectral expanders. That is:

**Theorem** (Oppenheim’s Trickling Down Theorem). Let \((X, \Pi)\) be a \( d \)-dimensional weighted simplicial complex satisfying the following two properties:

1. Every link of dimension \( d - 2 \) is a two-sided \( \gamma \)-spectral expander
2. Every link of dimension \( \leq d - 2 \) is connected.

Then \((X, \Pi)\) is a two-sided \( \frac{2}{1-(d-2)\gamma} \)-local spectral expander.

In fact, it’s possible to prove a stronger result: negative eigenvalues actually improve through trickle-down!

**Bonus Problem (no points):** Let \( X \) be a 3-dimensional complex. Prove that if the smallest negative eigenvalue across 1-links is \( \eta \), then the smallest negative eigenvalue of the graph underlying \( X \) is at least \( \frac{\eta}{1-\eta} > \eta \).

Notice that this lets us turn one-sided local-spectral expanders into two-sided local-spectral expanders just by truncating the complex at some dimension \( k \ll d \)!

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\(^1\)You may choose skip this part and assume the links are connected without proof

\(^2\)By definition, \( \sigma \) is a partial flag (a complete flag with some subspaces removed). A dimension is “missing” if its corresponding subspace has been removed.