LECTURE 16

LECTURE OUTLINE

- Approximation approach for convex optimization algorithms:
- Cutting plane method
- Simplicial decomposition

- Reading: Section 6.4 of on-line Chapter 6 on algorithms
CUTTING PLANE METHOD

- **Problem**: Minimize \( f : \mathbb{R}^n \to \mathbb{R} \) subject to \( x \in X \), where \( f \) is convex, and \( X \) is closed convex.

- **Method**: Start with any \( x_0 \in X \). For \( k \geq 0 \), set

\[
x_{k+1} \in \arg \min_{x \in X} F_k(x),
\]

where

\[
F_k(x) = \max \{ f(x_0) + (x-x_0)'g_0, \ldots, f(x_k) + (x-x_k)'g_k \}
\]

and \( g_i \) is a subgradient of \( f \) at \( x_i \).
CONVERGENCE OF CUTTING PLANE METHOD

\[ F_k(x) = \max \{ f(x_0) + (x - x_0)'g_0, \ldots, f(x_k) + (x - x_k)'g_k \} \]

\[ F_k(x_{k+1}) \leq F_k(x) \leq f(x), \quad \forall x \]

- \( F_k(x_k) \) increases monotonically with \( k \), and all limit points of \( \{x_k\} \) are optimal.

**Proof:** (Abbreviated) If \( x_k \to \bar{x} \) then \( F_k(x_k) \to f(\bar{x}) \), [otherwise there would exist a hyperplane strictly separating epi(\( f \)) and \( (\bar{x}, \lim_{k \to \infty} F_k(x_k)) \)]. This implies that

\[ f(\bar{x}) \leq \lim_{k \to \infty} F_k(x) \leq f(x), \quad \forall x \in X \]

Q.E.D.
TERMINATION

- We have for all $k$

$$F_k(x_{k+1}) \leq f^* \leq \min_{i \leq k} f(x_i)$$

- Termination when $\min_{i \leq k} f(x_i) - F_k(x_{k+1})$ comes to within some small tolerance.

- For $f$ polyhedral, we have finite termination with an exactly optimal solution.

- **Instability problem:** The method can make large moves that deteriorate the value of $f$.

- Starting from the exact minimum it typically moves away from that minimum.
VARIANTS

• **Variant I**: Simultaneously with $f$, construct polyhedral approximations to $X$.

• **Variant II**: Central cutting plane methods
SIMPLICIAL DECOMPOSITION IDEAS

- Minimize a differentiable convex $f : \mathbb{R}^n \rightarrow \mathbb{R}$ over bounded polyhedral constraint set $X$.
- Approximate $X$ with a simpler inner approximating polyhedral set.

![Diagram of a simplex and level sets of $f$]

- Approximating problem (min over a simplex):
  
  $$\begin{align*}
  \text{minimize} & \quad f \left( \sum_{j=1}^{k} \alpha_j \tilde{x}_j \right) \\
  \text{subject to} & \quad \sum_{j=1}^{k} \alpha_j = 1, \quad \alpha_j \geq 0
  \end{align*}$$

- Construct a more refined problem by solving a linear minimization over the original constraint.
• Given current iterate $x_k$, and finite set $X_k \subset X$ (initially $x_0 \in X$, $X_0 = \{x_0\}$).

• Let $\tilde{x}_{k+1}$ be extreme point of $X$ that solves

\[
\begin{align*}
\text{minimize} & \quad \nabla f(x_k)'(x - x_k) \\
\text{subject to} & \quad x \in X
\end{align*}
\]

and add $\tilde{x}_{k+1}$ to $X_k$: $X_{k+1} = \{\tilde{x}_{k+1}\} \cup X_k$.

• Generate $x_{k+1}$ as optimal solution of

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \text{conv}(X_{k+1}).
\end{align*}
\]
CONVERGENCE

- There are two possibilities for $\tilde{x}_{k+1}$:
  
  (a) We have

  $$0 \leq \nabla f(x_k)'(\tilde{x}_{k+1} - x_k) = \min_{x \in X} \nabla f(x_k)'(x - x_k)$$

  Then $x_k$ minimizes $f$ over $X$ (satisfies the optimality condition)

  (b) We have

  $$0 > \nabla f(x_k)'(\tilde{x}_{k+1} - x_k)$$

  Then $\tilde{x}_{k+1} \notin \text{conv}(X_k)$, since $x_k$ minimizes $f$ over $x \in \text{conv}(X_k)$, so that

  $$\nabla f(x_k)'(x - x_k) \geq 0, \quad \forall x \in \text{conv}(X_k)$$

- Case (b) cannot occur an infinite number of times ($\tilde{x}_{k+1} \notin X_k$ and $X$ has finitely many extreme points), so case (a) must eventually occur.

- The method will find a minimizer of $f$ over $X$ in a finite number of iterations.
The method is appealing under two conditions:

- Minimizing $f$ over the convex hull of a relative small number of extreme points is much simpler than minimizing $f$ over $X$.

- Minimizing a linear function over $X$ is much simpler than minimizing $f$ over $X$.

Important specialized applications relating to routing problems in data networks and transportation.
VARIANTS OF SIMPLICIAL DECOMP.

- Variant to remove the boundedness assumption on $X$ (impose artificial constraints).

- **Variant to enhance efficiency**: Discard some of the extreme points that seem unlikely to “participate” in the optimal solution, i.e., all $\tilde{x}$ such that

$$\nabla f(x_{k+1})' (\tilde{x} - x_{k+1}) > 0$$

- Additional methodological enhancements:
  - **Extension to $X$ nonpolyhedral** (method remains unchanged, but convergence proof is more complex)
  - **Extension to $f$ nondifferentiable** (requires use of subgradients in place of gradients, and more sophistication)
  - **Duality relation with cutting plane methods based on Fenchel duality.**

- We will derive, justify, and extend these by showing that cutting plane and simplicial decomposition are special cases of two polyhedral approximation methods that are dual to each other (next lecture).
GENERALIZED SIMPLICIAL DECOMPOSITION

- Consider minimization of $f(x) + c(x)$, over $x \in \mathbb{R}^n$, where $f$ and $c$ are closed proper convex
- Case where $f$ is differentiable

- Form $C_k$: inner linearization of $c$ [epi($C_k$) is the convex hull of the halflines $\{(\tilde{x}_j, w) \mid w \geq f(\tilde{x}_j)\}$, \(j = 1, \ldots, k\)]. Find
  
  $$x_k \in \arg \min_{x \in \mathbb{R}^n} \{f(x) + C_k(x)\}$$

- Obtain $\tilde{x}_{k+1}$ such that
  
  $$-\nabla f(x_k) \in \partial c(\tilde{x}_{k+1}),$$

and form $X_{k+1} = X_k \cup \{\tilde{x}_{k+1}\}$
NONDIFFERENTIABLE CASE

- Given $C_k$: inner linearization of $c$, obtain

$$x_k \in \arg \min_{x \in \mathbb{R}^n} \left\{ f(x) + C_k(x) \right\}$$

- Obtain a subgradient $g_k \in \partial f(x_k)$ such that

$$-g_k \in \partial C_k(x_k)$$

- Obtain $\tilde{x}_{k+1}$ such that

$$-g_k \in \partial c(\tilde{x}_{k+1})$$

and form $X_{k+1} = X_k \cup \{\tilde{x}_{k+1}\}$

- Example: $c$ is the indicator function of a polyhedral set