Assignment 2

Individual problems

1. (6 points)

(i) Consider a vocabulary with one binary relation $\text{CANFOOL}(p, t)$ intuitively meaning: “you can fool person $p$ at time $t$”. Write an FO sentence over this vocabulary expressing the following: You can fool some people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time. If you find ambiguities in this English statement, please explain which interpretation is reflected in your sentence.

(ii) Write an FO sentence over the vocabulary of directed graphs $(V, E)$ stating that the graph has diameter 3 (the diameter of a graph is the maximum distance between two nodes in the graph, where the distance is the number of edges in the shortest path between the two nodes).

(iii) Consider the FO vocabulary $(S, =, <)$, where $S$ is a unary relation (a set), and $<$ is a binary relation. Write an FO sentence $\varphi$ over this vocabulary so that, for each interpretation $I$ whose domain is the real numbers and that interprets $<$ in the standard way over the reals, $I \models \varphi$ iff $I$ interprets $S$ as a (finite or infinite) union of open intervals.

2. (3 points) Show that the compactness theorem for propositional logic can be proven using the compactness theorem for FO.

3. (2 points) Prove that there is no set $\Sigma$ of FO axioms (sentences) over vocabulary $(0, +, *, <)$ which is only satisfied by the real numbers (up to isomorphism). Does the choice of vocabulary make a difference? For example, would things change if the axioms are also allowed to talk about exponentiation? **Hint:** This looks hard, but is easy.
Group problems

4. (6 points) As we will see shortly, satisfiability of FO sentences is generally undecidable. We consider a syntactic subclass of FO for which satisfiability is decidable. Consider the class $\exists^* \forall^* \text{FO}$ of FO sentences of the form

$$\exists x_1 \ldots \exists x_k \forall y_1 \ldots \forall y_m \varphi$$

where $\varphi$ is a quantifier-free formula with relational vocabulary (no functions or constants).

(i) (5 points) Show that satisfiability of $\exists^* \forall^* \text{FO}$ sentences over relational vocabulary is decidable\(^1\).

(ii) (1 point) Using the above, exhibit a class of sentences for which validity is decidable.

5. (8 points) An FO sentence has the finite model property if either it has no model or it has at least one finite model.

(i) (1 point) Give an example of a sentence that has the finite-model property.

(ii) (3 points) Consider a vocabulary $\{f, =\}$, where $f$ is a unary function. Write an FO sentence $\varphi$ over this vocabulary that does not have the finite model property (i.e. $\varphi$ is satisfiable but has only infinite models).

(iii) (4 points) Show that for sentences that have the finite model property, satisfiability is decidable.

\(^1\)This also holds for arbitrary vocabularies but is a bit harder to show.