Matrix-Based Transforms

Image Processing
CSE 166
Lecture 12
Announcements

• Assignment 4 will be released May 15
• Reading
  – Chapter 6: Wavelet and Other Image Transforms
    • Section 6.2
Matrix-based transforms

\[ T(u) = \sum_{x=0}^{N-1} f(x)r(x, u) \quad \text{Forward transform} \]
\[ f(x) = \sum_{u=0}^{N-1} T(u)s(x, u) \quad \text{Inverse transform} \]

where

- \( x \) is a spatial variable
- \( u \) is a transform variable
- \( T(u) \) is the transform of \( f(x) \)
- \( f(x) \) is the inverse transform of \( T(u) \)
- \( r(x, u) \) is a forward transformation kernel
- \( s(x, u) \) is an inverse transformation kernel
General inverse transform using basis vectors

\[ f(x) = T(0)s(x, 0) + T(1)s(x, 1) + \ldots + T(N-1)s(x, N-1) \]
Matrix-based transforms using orthonormal basis vectors

• In vector form

\[ T(u) = \langle s(x, u), f(x) \rangle \]
\[ T(u) = \langle s_u, f \rangle \]

where \( s_u = \begin{bmatrix} s(0, u) \\ s(1, u) \\ \vdots \\ s(N - 1, u) \end{bmatrix} \) and \( f = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N - 1) \end{bmatrix} \)

\[ T(u) = s_u^H f \text{ for complex vectors} \]
\[ T(u) = s_u^T f \text{ for real vectors} \]
Matrix-based transforms using orthonormal basis vectors

• In matrix form

\[
\begin{bmatrix}
T(0) \\
T(1) \\
\vdots \\
T(N - 1)
\end{bmatrix}
= \begin{bmatrix}
    s_0^H \\
    s_1^H \\
    \vdots \\
    s_{N-1}^H
\end{bmatrix}
\begin{bmatrix}
    f(0) \\
    f(1) \\
    \vdots \\
    f(N - 1)
\end{bmatrix}
\]

for complex vectors

\[
\begin{bmatrix}
T(0) \\
T(1) \\
\vdots \\
T(N - 1)
\end{bmatrix}
= \begin{bmatrix}
    s_0^T \\
    s_1^T \\
    \vdots \\
    s_{N-1}^T
\end{bmatrix}
\begin{bmatrix}
    f(0) \\
    f(1) \\
    \vdots \\
    f(N - 1)
\end{bmatrix}
\]

for real vectors

\[t = Af\]
\[f = A^H t \text{ for complex vectors}\]
\[f = A^T t \text{ for real vectors}\]
Matrix-based transforms using biorthonormal basis vectors

- In matrix form

\[
\begin{bmatrix}
T(0) \\
T(1) \\
\vdots \\
T(N-1)
\end{bmatrix} = \begin{bmatrix}
\tilde{s}_0^H \\
\tilde{s}_1^H \\
\vdots \\
\tilde{s}_{N-1}^H
\end{bmatrix} \begin{bmatrix}
f(0) \\
f(1) \\
\vdots \\
f(N-1)
\end{bmatrix}
\]

for complex vectors

\[
\begin{bmatrix}
T(0) \\
T(1) \\
\vdots \\
T(N-1)
\end{bmatrix} = \begin{bmatrix}
\tilde{s}_0^T \\
\tilde{s}_1^T \\
\vdots \\
\tilde{s}_{N-1}^T
\end{bmatrix} \begin{bmatrix}
f(0) \\
f(1) \\
\vdots \\
f(N-1)
\end{bmatrix}
\]

for real vectors

\[t = \tilde{A}f\]

\[f = A^H t \text{ for complex vectors}\]

\[f = A^T t \text{ for real vectors}\]
Matrix-based transform

Example: 8-point DFT of $f(x) = \sin(2\pi x)$

real part + imaginary part
Matrix-based transforms in two dimensions

\[ T(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) r(x, y, u, v) \quad \text{Forward transform} \]

\[ f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} T(u, v) s(x, y, u, v) \quad \text{Inverse transform} \]

where

- \( x, y \) are spatial variables
- \( u, v \) are transform variables
- \( T(u, v) \) is the transform of \( f(x, y) \)
- \( f(x, y) \) is the inverse transform of \( T(u, v) \)
- \( r(x, y, u, v) \) is a forward transformation kernel
- \( s(x, y, u, v) \) is an inverse transformation kernel
Matrix-based transforms in two dimensions

• If $r$ and $s$ are separable and symmetric, and $M = N$, then
  
  – For orthonormal basis vectors
    
    \[
    T = AFA^T
    \]
    
    \[
    F = A^*TA^* \text{ for complex vectors}
    \]
    
    \[
    F = A^T TA \text{ for real vectors}
    \]
  
  – For biorthonormal basis vectors
    
    \[
    T = \tilde{A}\tilde{F}\tilde{A}^T
    \]
    
    \[
    F = A^*TA^* \text{ for complex vectors}
    \]
    
    \[
    F = A^T TA \text{ for real vectors}
    \]
Next Lecture

• Basis images
• Wavelet transforms
• Reading
  – Chapter 6: Wavelet and Other Image Transforms
    • Sections 6.5 and 6.10