Instructions Upload a single file to Gradescope for each group. All group members’ names and PIDs should be on each page of the submission. Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Reading Sipser Sections 3.1, 3.2, 4.1

Key Concepts TM, Decidability
Recall the terminology for describing Turing Machines as per the class slides:

- **Formal definition**: set of states, input alphabet, tape alphabet, transition function, state state, accept state, reject state. A state diagram is sufficient for defining the transition function.
- **Implementation-level definition**: English prose to describe Turing machine head movements relative to contents of tape.
- **High-level description**: Description of algorithm, without implementation details of machine. As part of this description, can "call" and run another TM as a subroutine.

1. (10 points) For a string $s \in \{0, 1, 2\}^*$ and a symbol $d \in \{0, 1, 2\}$ let $\#(s, d)$ denote the number of times $d$ appears in $s$. For example, $\#(0120012, 0) = 3$. Consider the language:

   $$ L = \{ u\#w \mid u, w \in \{0, 1, 2\}^*, \#(u, 0) \leq \#(w, 0), \#(u, 1) \leq \#(w, 1), \#(u, 2) \leq \#(w, 2) \}. $$

   For example, $201202\#0011222 \in L$.

   Construct a TM that decides this language. Provide a **formal definition** of your TM.

2. (10 points) For a string $s \in \{0, 1\}^*$ let $s_2$ denote the number represented by $s$ in the binary numeral system. For example, $1110$ in binary has a value of $14$. Consider the language:

   $$ L = \{ u\#w \mid u, w \in \{0, 1\}^*, u_2 + 1 = w_2 \}, $$

   meaning it contains all strings $u\#w$ such that $u + 1 = w$ holds true in the binary system. For example, $1010\#1011 \in L$ and $0011\#100 \in L$. Construct a TM that decides this language.

   Provide an **implementation-level definition** of your TM.

3. (10 points) Consider a pushdown automaton (PDA) with 2 stacks, where both stacks use the same alphabet $\Gamma$ and the transition function is of the following form $\delta : Q \times \Sigma_e \times \Gamma^2 \rightarrow \mathcal{P}(Q \times \Gamma^2)$. Show that PDA with 2 stacks can simulate any TM. (Hint: show how to use the two stacks of PDA to simulate a tape of a TM.) Provide an **implementation-level definition** of your PDA.

4. (10 points)
   (a) Show that the class of decidable languages is closed under complement.
   (b) Can you show that Turing-recognizable languages are also closed under complement? If not, explain the problem - where does the proof break?

   Provide a high-level description of any TMs you construct.

5. (10 points) Define a **de-numerator** of a language $L$ as a machine that prints all the strings not in $L$ in an order of **non-decreasing length**. Prove a language is Turing decidable if it has a de-numerator. (Hint: consider the two disjoint cases, first when the complement of $L$ is finite and second when it is infinite. In the first case you can show $L$ is decidable without the de-numerator.) Provide a high-level description of any TMs you construct.