Summary

Last time:

- Logical connectives: “not”, “and”, “or”, “implies”
- Using Truth Tables to define logical connectives

Today:

- Logical equivalences, tautologies
- Some applications
- Proofs in propositional logic
- Reading: Chap. 1.1, 1.2, 1.3, 1.6
“I will get an A in CSE20, **unless** I get sick”

A = “I will get an A in CSE20”; B = “I get sick”

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“A unless B” has the same meaning as

(A) $A \land B$; (B) $A \rightarrow B$; (C) $\neg(\neg A \land \neg B)$;

(D) $A \lor B$; (E) None of the above


“\(A\) unless \(B\)” has the same meaning as

- \((A)\) \(A \land B\); \((B)\) \(A \rightarrow B\); \((C)\) \(\neg (\neg A \land \neg B)\);
- \((D)\) \(A \lor B\); \((E)\) None of the above

- \(\neg (\neg A \land \neg B)\): false if don’t get sick, and still don’t get an A
- \(A \lor B\): either I get an A or I get sick (or both)
Conditional statements in English

Many equivalent ways to express the implication $p \rightarrow q$ in English:

“If I become Chancellor, I will lower tuition and fees.”

- if $p$ then $q$
- $q$ if $p$
- $q$ when $p$
- $p$ is a **sufficient condition** for $q$
- $q$ is a **necessary condition** for $p$
- $q$ unless $\neg p$
- $p$ implies $q$
- $p$ only if $q$
- $q$ follows from $p$
Many equivalent ways to express the implication $p \rightarrow q$ in English:

“If I become Chancellor, I will lower tuition and fees.”

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Double implication $(p \leftrightarrow q) \equiv (p \rightarrow q) \land (q \rightarrow p)$

- $p$ if and only if $q$
- $p$ is necessary and sufficient for $q$
Claim: \( \neg A \land \neg B \equiv \neg (A \lor B) \)

Proof: We evaluate \( \neg A \land \neg B \) and \( \neg (A \lor B) \) by computing the truth value of each subexpression.
Claim:  \( \neg A \land \neg B \equiv \neg (A \lor B) \)

Proof:  We evaluate \( \neg A \land \neg B \) and \( \neg (A \lor B) \) by computing the truth value of each subexpression.

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The columns for \( \neg A \land \neg B \) and \( \neg (A \lor B) \) are identical.

\( \neg (A \lor B) \) and \( \neg A \land \neg B \) are logically equivalent.

“The negation of a disjunction is the conjunction of the negations”
Logical identities

- De Morgan (and): $\neg(A \lor B) \equiv \neg A \land \neg B$
- De Morgan (or): $\neg(A \land B) \equiv \neg A \lor \neg B$
- Double negation: $\neg(\neg A) \equiv A$
- Commutativity (and): $A \land B \equiv B \land A$
- Commutativity (or): $A \lor B \equiv B \lor A$
- Associativity (and): $A \land (B \land C) \equiv (A \land B) \land C$
- Associativity (or): $A \lor (B \lor C) \equiv (A \lor B) \lor C$
- Distributivity
- ...

All very useful to simplify logical expressions:

$\neg(\neg A \land \neg B) \equiv \neg\neg A \lor \neg\neg B \equiv A \lor B$
What does $\neg p \lor q$ mean?

- (A) $(\neg p) \lor q$
- (B) $\neg(p \lor q)$
- (C) Both, they are logically equivalent
- (D) It depends on the choice of $p$ and $q$. 

The standard answer is (A), but the choice is purely conventional.

Similarly, you may ask if $p \rightarrow q \rightarrow r$ means $p \rightarrow (q \rightarrow r)$ or $(p \rightarrow q) \rightarrow r$. (“right” or “left” associativity.)
What does $\neg p \vee q$ mean?

- (A) $(\neg p) \vee q$
- (B) $\neg(p \vee q)$
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The standard answer is (A), but the choice is purely conventional.

Similarly, you may ask if $p \rightarrow q \rightarrow r$ means $p \rightarrow (q \rightarrow r)$ or $(p \rightarrow q) \rightarrow r$. (“right” or “left” associativity.)
From highest to lower precedence:

- ¬
- ∧
- ∨
- →
- ↔

So, \( p \land q \rightarrow p \lor \neg q \land p \) means So, \((p \land q) \rightarrow (p \lor ((\neg q) \land p))\).

Similar to usual arithmetic precedence rules:

- \( 5 \cdot 6^2 + 4 \) means \((5 \cdot (6^2)) + 4\).

\( p \rightarrow q \) and \( x^y \) are usually considered right associative:

\( p \rightarrow q \rightarrow r \equiv p \rightarrow (q \rightarrow r) \), \( 3^{4^5} \equiv 3^{(4^5)} \).
Conditional “if $X$ then $Y$ else $Z$”

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Question: How many rows in the truth table?

(A) 4  (B) 6  (C) 8  (D) 9  (E) None of the above
Truth table for “if $X$ then $Y$ else $Z$”

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Answer: (C)
Can you express “if X then Y else Z” in terms of the other connectives?

- **(A)** \((X \rightarrow Y) \land (\neg X \rightarrow Z)\)
- **(B)** \((X \land Y) \lor (\neg X \land Z)\)
- **(C)** These are both valid answers
- **(D)** None of the above
Can you express “if $X$ then $Y$ else $Z$” in terms of the other connectives?

- **(A)** $(X \rightarrow Y) \land (\neg X \rightarrow Z)$
- **(B)** $(X \land Y) \lor (\neg X \land Z)$
- **(C)** These are both valid answers
- **(D)** None of the above

I think the correct answer is (B).

How can we check?
Can you express “if $X$ then $Y$ else $Z$” in terms of the other connectives?

- (A) $(X \rightarrow Y) \land (\neg X \rightarrow Z)$
- (B) $(X \land Y) \lor (\neg X \land Z)$
- (C) These are both valid answers
- (D) None of the above

I think the correct answer is (B).

How can we check?

Let’s draw a truth table
A logical statement is a tautology if it is always true.

Example: Law of excluded middle ("tertium non datur", there is no third possibility)

\( A \lor \neg A \)

Prove using the truth table method:

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<th>( \neg A )</th>
<th>( A \lor \neg A )</th>
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<tr>
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Last column is always true.
Another tautology

“Modus ponens”

\[ ((A \rightarrow B) \land A) \rightarrow B \]

- If \( A \) implies \( B \)
- and \( A \) is true
- then \( B \) is also necessarily true

Let’s check it by drawing a truth table.
Transitivity of implication.

\(((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C)\)

If A implies B and B implies C, then A implies C.

Question: How many rows in the truth table?

(A) 6; (B) 4; (C) 8; (D) 16
Transitivity of implication.

\[ ((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C) \]

If \( A \) implies \( B \) and \( B \) implies \( C \), then \( A \) implies \( C \).

Question: How many rows in the truth table?

(A) 6; (B) 4; (C) 8; (D) 16

Answer: (C) \( 2^3 = 8 \). There are 3 propositional variables \((A, B, C)\), and each can take two possible values.
Transitivity of implication.

\[ ((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C) \]

If \( A \) implies \( B \) and \( B \) implies \( C \), then \( A \) implies \( C \).

Question: How many rows in the truth table?

(A) 6; (B) 4; (C) 8; (D) 16

Answer: (C) \( 2^3 = 8 \). There are 3 propositional variables \((A, B, C)\), and each can take two possible values.

Question: How many columns?

(A) 4; (B) 7; (C) 8; (D) 11
One more tautology

Transitivity of implication.

\[ ((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C) \]

If \( A \) implies \( B \) and \( B \) implies \( C \), then \( A \) implies \( C \).

Question: How many rows in the truth table?

(A) 6; (B) 4; (C) 8; (D) 16

Answer: (C) \( 2^3 = 8 \). There are 3 propositional variables \((A, B, C)\), and each can take two possible values.

Question: How many columns?

(A) 4; (B) 7; (C) 8; (D) 11

Answer: \( A, B, C, A \rightarrow B, B \rightarrow C, A \rightarrow C, (A \rightarrow B) \land (B \rightarrow C), ((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C) \).
Tautologies and equivalences

Double implication: \((A \leftrightarrow B) \equiv ((A \rightarrow B) \land (B \rightarrow A))\)

Two (compound) propositional formulas \(P, Q\) are equivalent \((P \equiv Q)\) if and only if \((P \leftrightarrow Q)\) is a tautology.

Terminology:

- \(P\) is a **tautology** if it is always true
- \(P\) is a **contradiction** if it is always false
- \(P\) is a **contingency** if it can be both true or false